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DYNAMICS OF SATELLITE WIRE-BOOM SYSTEMS

Shu T. Lai, et al

Logicon, Incorporated

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21535 Hawthorne Boulevard
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This is a theoretical analysis and computer simulation problem involving the mathematical formulation, derivation and computation of all mode characteristics of a coupled spinning satellite hub-wire boom system; the digital simulation generates time-dependent dynamical behavior of the satellite system under different experimental conditions.

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CHAPTER 1

INTRODUCTION

In one of the Air Force Cambridge Research Laboratories programs to study the ionosphere, satellite experiments with wire booms have been devised to measure and model the earth's ambient electric field. The wire booms serve as sensors, an electric field being induced in them as the satellite traverses the earth's magnetic field in its orbit. A signal proportional to the unknown ambient electric field is also summed in vectorially in the measurement:

$$E_{measurement} = (\underline{V} \times \underline{B}) \cdot \underline{R} + \underline{E}_{ambient} \cdot \underline{R}$$

where $\underline{V}$ is the orbital velocity of the sensor of satellite with respect to the magnetic field $\underline{B}$, and $\underline{R}$ is the sensor position vector.

In order to extend the wire booms properly and to maintain a stable orientation for the experiment, a spin is imparted to the satellite. The vector equation above still holds at every instant, and the desired ambient field may be calculated provided the wire boom lengths and orientations are known as a function of time. Attitude sensors on the spacecraft provide information about the average motion of the vehicle, but the superimposed perturbations and the orientation of the booms themselves are less readily resolved.

This report addresses this problem by developing analytical and simulated models of the satellite and wire boom dynamics. Experimental observations may thereby be correlated with expected behavior during the data reduction and evaluation phase. Control information during operational conditions is also derivable for decisions such as extent and times of boom retraction or deployment. Earlier works on different spacecrafts have reported on studies of electric field and ion measurement as a function of sensor and vehicle attitude [4, 15].

This study is initially intended to support the research experiment CRL-22e on satellite 1975, to be conducted by the Electrical Processes Branch of AFCRL's Space Physics Laboratory, for the global measurement of electric fields in the ionospheric region. Feasibility studies, mass properties, and design of 1975 have been reported by Boeing [1, 2].

The satellite has a relatively heavy hub (490.9 lb) with a shape resembling a rectangular box, with several sensors on rigid or flexible booms. Four of the booms are flexible, each carrying a tip mass (2 lb), and are deployable

pairwise. An opposite pair of booms would have unequal lengths only as a very unlikely event. A wobble damper in designed to reduce as much as possible any out-of-plane motion, and a Coulomb damper is also built in to damp excessive in-plane boom oscillation. The hub spin is usually about three RPM. The mechanisms of deployment of booms and spinning of satellite are controllable from the ground. More details on engineering aspects are given in Chapter 8.

A substantial flow of analyses has covered various aspects of satellite-boom dynamics in the last decade. Examples are a study of spin dynamics to determine stresses in rigid booms [8], and a study of maximum nutation-precession angles, bending moments, and deflections due to boom deployment [5]. In contrast, this study emphasizes the deployable taut wire booms, and attempts to predict mode frequencies, damping, satellite spin, and boom deflections. Nutation-precession is assumed to be minimized due to the wobble damper.

The dynamics of the 1975 satellite system are composed of a variety of modes: coupled vehicle-boom oscillations, boom vibrations, translation, and precession about the equilibrium position, in addition to orbital motion and self-spinning. Effects of aerodynamics are small at the altitude of the satellite orbit (see Chapter 8). Boom vibrations have been determined to be insignificant compared to the pendulum type coupled hub-boom oscillations [17]. Solar radiation induced oscillations and bending of wire booms is considered to be insignificant for thin wires, even though this solar effect has been experienced before in spacecrafts with tubular cables of lengths of the order of a kilometer [6, 9]. Earth's gravitational gradient induced oscillations are also negligible in view of the short boom lengths (0 to 60 ft.) [7]. In practice, the signal measured in the wire booms will be affected by a number of other factors such as shadow effects, satellite wake effects, vehicle potential, and instrumentation effects. A systematic isolation of these factors is part of the data reduction and evaluation logic, and is outside the scope of this report.

This theoretical analysis includes the formulation, derivation, and computation of all mode characteristics of coupled hub-boom dynamics. The computer programs written are capable of generating digital simulations of time-dependent dynamical behavior of the satellite system under different experimental condition. Schematically, this work is summarized in t' flow chart in Figure 1.

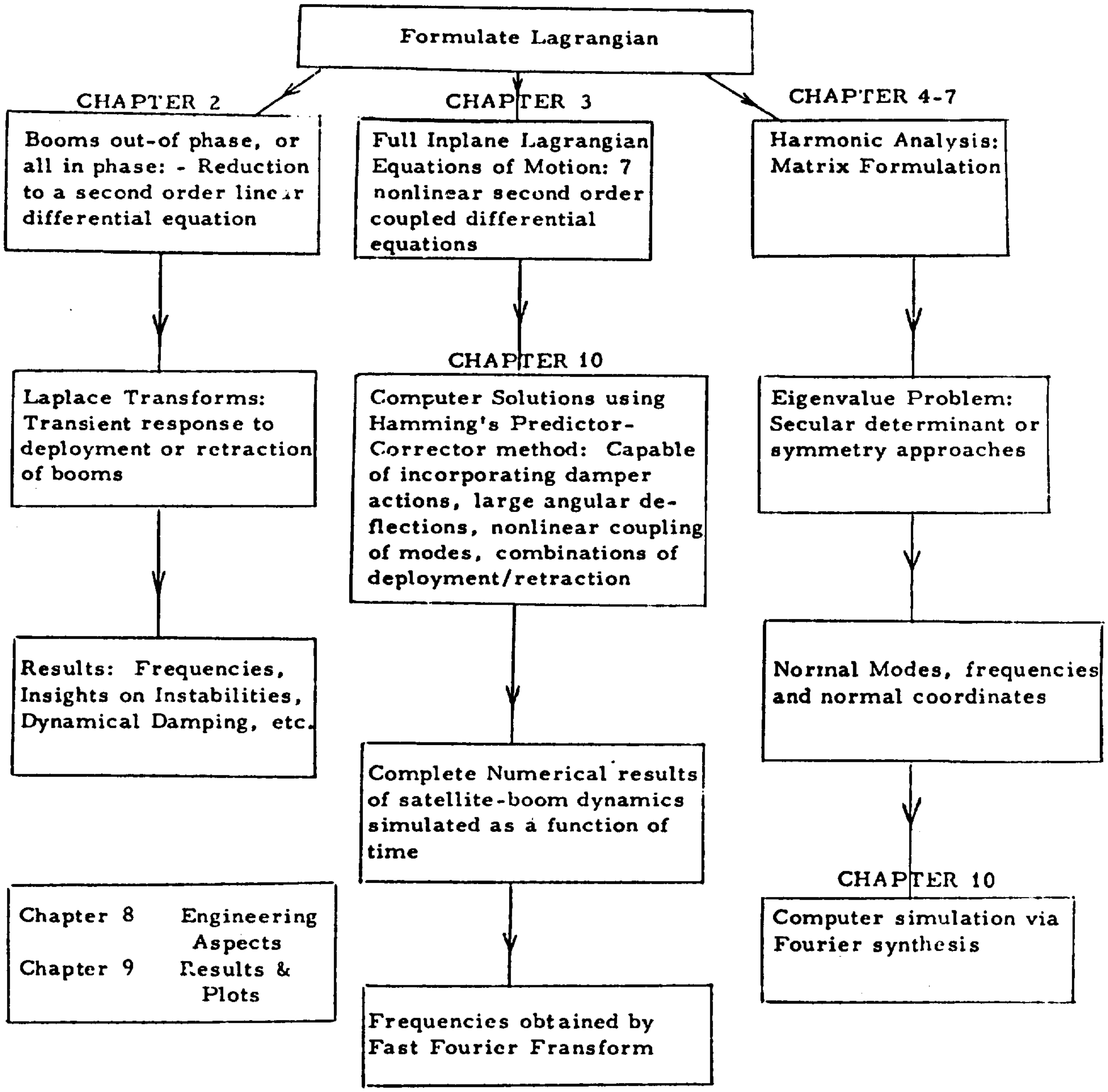


Figure 1. Analysis Procedures for Satellite-Boom Dynamics

Transient response of the system and insight into instabilities and dynamical damping properties are derived using Laplace transform techniques on a linearized set of equations of motion. This preliminary study is restricted to the simpler symmetrical coupled or uncoupled cases, and is covered in Chapter 2.

For satellite system dynamics in the spin plane, with taut wire-booms, seven degrees of freedom exist viz. hub spin rate, hub X and Y translations, and angular deviation of each wire boom from its normal radial direction. Deployment of each boom is specified. The complete Lagrangian equations of motion are developed for this system in Chapter 3, and result in seven non-linear second order coupled differential equations. A complete computer solution of these equations including special damper actions, arbitrary deflection amplitudes, and deployment/retraction of wire booms was implemented and is described in Chapter 10, the section on program documentation. Characteristic frequencies are obtained by Fast Fourier Transform of the dynamic responses as a function of time.

In general, it is desirable to estimate all mode characteristics without resorting to the solution and Fourier analysis of the complete Lagrangian equations. Further, in-plane/out-of-plane interacting dynamical equations become algebraically too complicated and are not attempted. Instead, eigenvalue techniques and orthonormal matrix transformations are used to derive all the mode characteristics, both in and normal to the hub spin plane. Matrices are of size 14x14 corresponding to the total number of degrees of freedom of the system, however assuming minimal in-plane/out-of-plane interaction, they are decomposed into two sets of 7x7 matrices.

The in-plane analytic solution is carried out in Chapter 4. In terms of the deviations η_i in generalized coordinates q_i where $q_i = q_{oi} + \eta_i$, the kinetic energy T and potential energy V are expanded in Taylor series. For harmonic motions about the equilibrium configurations, terms of third and higher order in η_i do not contribute. As a result, the Lagrangian is given by

$$L = \frac{1}{2} \sum_{i,j} \left(T_{ij} \dot{\eta}_i \dot{\eta}_j - V_{ij} \eta_i \eta_j \right)$$

where the dot denotes time derivatives. The Lagrangian leads to the following equations of motion:

$$\Sigma_j T_{ij} \ddot{\eta}_j + \Sigma_j V_{ij} \eta_j = 0$$

For nontrivial harmonic motion to exist, the secular determinantal condition must be satisfied:

$$\det \left| V_{ij} - \omega^2 T_{ij} \right| = 0$$

which yields the eigenvalues ω_i^2, and the eigenfunctions.

Alternately, a new set of coordinates ζ_i, so called normal coordinates, can be sought as in Chapters 5 - 7 such that

$$\eta_i = \sum_j B_{ij} \zeta_j$$

and

$$T = \frac{1}{2} \sum_j \dot{\zeta}_j^2$$

$$V = \frac{1}{2} \sum_j \omega_j^2 \zeta_j^2$$

This is an algebraic process of the simultaneous diagonalization of two quadratic forms. The resulting Lagrangian becomes simple and elegant:

$$L = \frac{1}{2} \sum_j (\dot{\zeta}_j^2 - \omega_j^2 \zeta_j^2)$$

From this it is well known [11] that ω_j^2 are the eigenvalues, yielding the desired characteristic frequencies.

Chapter 5 covers the case of inplane normal coordinates for equal boom lengths, and includes translation. Seven eigenfrequencies are derived corresponding to the seven degrees of freedom. The symmetrical cases that exclude translation, and for boom wire mass negligible compared to boom tip mass, yield the familiar results:

$$\omega = \omega_o \sqrt{\frac{r_o}{r}} \qquad \text{(uncoupled)}$$

$$\omega = \omega_o \sqrt{\frac{r_o}{r} \frac{I_T}{I_o}} \qquad \text{(coupled)}$$

where I_T is the total moment of inertia. The uncoupled mode is triply degenerate. The trivial modes corresponding to pure rotation and pure translation have been eliminated.

Out-of-plane normal coordinates are considered in Chapter 6, and are limited to cases of equal length booms and negligible interaction with in-plane frequencies. There are again seven degrees of freedom leading to seven modes, three of which - pure rotation and pure translation - are trivial.

Each of the normal coordinates is a periodic function involving only one of the resonance frequencies of the system. The frequencies are found as:

$$\omega_8 = \omega_9 = \omega_{10} = 0$$

$$\left.\begin{aligned} \omega_{11} &= \omega_0 \sqrt{\frac{r + r_0}{r} \quad \frac{I_{1T}}{I_{10}}} \\ \omega_{12} &= \omega_{11} \end{aligned}\right\} \quad \text{Coupled Modes}$$

$$\left.\omega_{13} = \omega_0 \sqrt{\frac{r + r_0}{r}}\right\} \quad \text{Uncoupled Saddle Mode}$$

$$\left.\omega_{14} = \omega_0 \sqrt{\frac{r + r_0}{r} \quad \frac{M + 4m}{M}}\right\} \quad \text{Jelly-Fish Mode}$$

These frequencies, together with the seven in-plane ones, comprise a total of 14 normal mode frequencies of the entire satellite-boom sensor system.

In plane oscillations with unequal length booms completes the analytical work. This is covered in Chapter 7, again with the use of normal coordinates. Some translation of the hub can be expected for all unequal boom length cases, unless the opposite booms are symmetrically deployed. Thus, in general, four distinct nontrivial eigenfrequencies can be found. The behavior of these ω-roots are revealed by plotting.

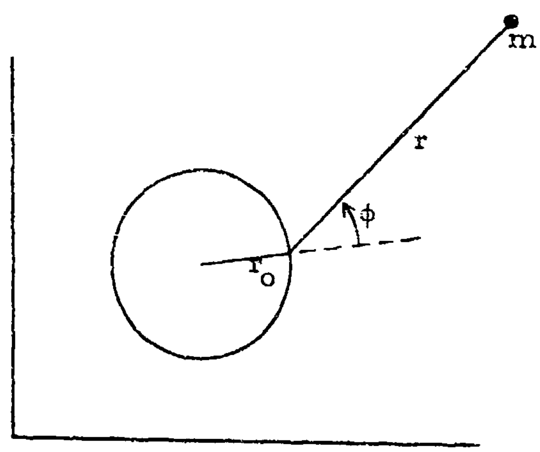
Chapter 8 establishes the background for obtaining the physical quantities of the satellite-boom system pertinent to this dynamics study. Chapter 9 presents typical results and plots. A number of support programs were required, but the main programs only are covered in Chapter 10. These include digital simulation of the satellite dynamics using the complete Lagrangian equations including external and inherent nonlinearities; the subsequent Fast Fourier Transform to obtain characteristic frequencies; and a scheme for synthesizing dynamic response from normal modes with specified initial conditions. Appendices A through H support various derivations in the text of the analysis.

CHAPTER 2

LINEARIZED RESPONSE OF THE SATELLITE-BOOM SYSTEM

2.1 Effect of Deployment or Retraction

In order to obtain physical insights on the effects or deployment or retraction, it is useful to study a simple but analytically tractable model: the one-boom satellite with no translation. The full-fledged coupled four-boom vibrating-translating satellite will be studied in later chapters. Both uncoupled and coupled modes will be treated in this chapter. Laplace transform techniques will be employed with the assumption of small boom length variation, i.e., $\dot{r}\,\Delta t/r \ll 1$.



To formulate the problem, one starts by writing down the coordinates, velocities, and energies of the hub, tip mass, and wire boom in terms of inertial coordinates. Then a transformation from inertial coordinates to corotating polar coordinates (r, ϕ) is facilitated by:

$$\dot{\underline{r}} \longrightarrow \dot{\underline{r}} + \underline{\omega}_o \times \underline{r}$$

The Lagrangian is then obtained, from which all the dynamics can be unfolded.

The Lagrangian L of the one boom system is

$$L = \frac{1}{2} I_o \dot{\theta}^2 + \frac{1}{2}(m + \rho r)\dot{r}^2 + \frac{1}{2}\left(mr^2 + \rho\frac{r^3}{3}\right)\dot{\phi}^2 + \omega\left[\left(mr^2 + \rho\frac{r^3}{3}\right)\dot{\phi}\right.$$

$$\left. + (m + \rho r)\, r_o\, \dot{r} \sin\phi + \left(mr + \rho\frac{r^2}{2}\right) r_o \cos\phi\, \dot{\phi}\right] + \frac{1}{2}\omega^2\left[\left(mr^2 + \rho\frac{r^3}{3}\right)\right.$$

$$+ (m + \rho r)\, r_o^2 + 2(mr + \rho \frac{r^2}{2})\, r_o \cos\phi] - \frac{1}{2} s \phi^2 \qquad (2\text{-}1)$$

The r equation of motion gives the tension on the boom wire. The ϕ equation of motion is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = - k_\phi \dot{\phi}$$

i. e.

$$(mr^2 + \rho\frac{r^3}{3})\ddot{\phi} + 2\dot{r}\,(mr + \rho\frac{r^2}{2})\dot{\phi} + \dot{\omega}[\,(mr^2 + \rho\frac{r^3}{3}) +$$

$$(mr + \rho\frac{r^2}{2})\, r_o \cos\phi\,] + \omega^2 (mr + \rho\frac{r^2}{2})\; r_o \sin\phi + 2\omega\,\cdot$$

$$\cdot (mr + \rho\frac{r^2}{2})\,\dot{r} = - k_\phi \dot{\phi} - s\phi \qquad (2\text{-}2)$$

In the limit $\rho \to 0$, ϕ equation becomes simplified to

$$mr^2\ddot{\phi} + 2mr\dot{r}\dot{\phi} + \ddot{\theta}\,[\,mr\,(r + r_o)] + \dot{\theta}^2 mr r_o \phi + 2mr\dot{r}\dot{\theta} + k_\phi \dot{\phi} +$$

$$+ s\phi = 0$$

where $\sin\phi \sim \phi$ and $\cos\phi \sim 1 - \phi^2/2 \sim 1$

The θ - equation of motion is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

where $\frac{\partial L}{\partial \theta} = - s\phi$

$$\frac{\partial L}{\partial \dot{\theta}} = I_T \dot{\theta} + (mr^2 + \rho\frac{r^3}{3})\dot{\phi} + (m + \rho r) r_o \dot{r} \sin\phi +$$

$$(mr + \rho\frac{r^2}{2})\, r_o \cos\phi\, \dot{\phi}$$

so that

$$I_T \ddot{\theta} = -\left[(mr^2 + \rho \frac{r^3}{3}) + (mr + \rho \frac{r^2}{2}) r_o \cos\phi \right] \ddot{\phi}$$

$$-2 \left[(mr + \rho \frac{r^2}{2}) + m + \rho r) r_o \cos\phi \right] \dot{r} \dot{\phi}$$

$$- \dot{I}_T \dot{\theta} + (mr + \rho \frac{r^2}{2}) r_o \sin\phi \dot{\phi}^2 - \rho r_o \dot{r}^2 \sin\phi - s\phi$$

For small amplitide oscillation, light boom wire, and small spring constant, the last three terms on the RHS are neglegible. For simplicity, we keep only the leading terms:

$$\ddot{\theta} = -\frac{1}{I_T} \left[(mr^2 + \rho \frac{r^3}{3}) + (mr + \rho \frac{r^2}{2}) r_o \cos\phi \right] \ddot{\phi}$$

$$-\frac{2}{I_T} \left[(mr + \rho \frac{r^2}{2}) + (m + \rho r) r_o \cos\phi \right] \dot{r} \dot{\phi} - \frac{\dot{I}_T}{I_T} \dot{\theta} \qquad (2\text{-}3)$$

$$\lim_{\rho \to 0} \ddot{\theta} = - \frac{mr\,(r + r_o)}{I_T} \ddot{\phi} - \frac{2m\,(r + r_o)\, \dot{r}}{I_T} \dot{\phi} - \frac{\dot{I}_T}{I_T} \dot{\theta} + \ldots$$

where the first term on the RHS is entirely due to boom vibration, the second and third terms are functions of boom deployment, and higher order terms are neglected. The result can also be obtained by considering conservation of angular momentum of the whole system.

In case of multiple booms, they may behave in such a way that their angular momenta add up to zero, so that the angular momentum of each boom is uncoupled to that of the hub: we call such a state an uncoupled mode. Thus,

$$\ddot{\theta} = \dot{\omega} = - \frac{\dot{I}_T}{I_T} \dot{\theta} \qquad \text{(uncoupled)}$$

and

$$\ddot{\theta} = \dot{\omega} = - \frac{\dot{I}_T}{I_T} \dot{\theta} - \frac{mr\,(r + r_o)}{I_T} \ddot{\phi} - \frac{2m\,(r + r_o)\, \dot{r}}{I_T} \dot{\phi} \qquad \text{(coupled)}$$

where

$$\dot{I}_T = 2 \left[(mr + \rho \frac{r^2}{2}) + \frac{\rho}{2} r_o^{\,2} + (m + \rho r)\, r_o \cos\phi \right] \dot{r}$$

$$- 2\,(mr + \rho \frac{r^2}{2})\, r_o \sin\phi\, \dot{\phi}$$

$$\lim_{\rho \to 0} \dot{I}_T = 2m\,(r + r_o)\dot{r} + \ldots$$

We remark that in the N-boom case the uncoupled $\ddot{\theta}$ is unchanged because the angular momenta of the booms cancel each other, whereas the coupled $\ddot{\theta}$ equation is obtained by assuming m to be replaced by Nm because all ϕ 's are in phase. Hence, provided there is no translation, the results of this chapter hold for multiple boom cases.

2.2 Forced Oscillation Equations

Substituting the above uncoupled $\ddot{\theta}$ and the coupled $\ddot{\theta}$ into the ϕ - equation of motion, one finds equations of motion in variable ϕ with θ eliminated.

For uncoupled oscillation mode, one finds:

$$\ddot{\phi}\,(mr^2 + \rho\frac{r^3}{3}) + \dot{\phi}\,2\dot{r}\,(mr + \rho\frac{r^2}{2}) - \frac{2\omega}{I_T}\left\{\dot{r}\,[\,mr + \rho\frac{r^2}{2} + \frac{\rho}{2}r_o^{\,2} + (m + \rho r)\,r_o\cos\phi\,] - (mr + \rho\frac{r^2}{2})\,r_o\sin\phi\,\dot{\phi}\right\}\cdot$$

$$\cdot\left\{mr^2 + \rho\frac{r^3}{3} + (mr + \rho\frac{r^2}{2})\,r_o\cos\phi\right\} + \omega^2\,(mr + \rho\frac{r^2}{2})\,r_o\sin\phi$$

$$+ 2\omega\,(mr + \rho\frac{r^2}{2})\,\dot{r} + k\dot{\phi} + s\phi = 0$$

For small ϕ, we have $\sin\phi \sim \phi$, $\cos\phi \sim 1$, $\phi\dot{\phi} \sim 0$ so that the above equation becomes

$$\ddot{\phi}\,(mr^2 + \rho\frac{r^3}{3}) + \dot{\phi}\,[\,2\dot{r}\,(mr + \rho\frac{r^2}{2}) + k\,] +$$

$$\phi\,[\,\omega^2\,(mr + \rho\frac{r^2}{2})\,r_o + s\,]$$

$$= -2\omega\dot{r}\left\{(mr + \rho\frac{r^2}{2}) - \frac{1}{I_T}\,[\,mr^2 + \rho\frac{r^3}{3} + (mr + \rho\frac{r^2}{2})\,r_o\,]\cdot\right.$$

$$\left.\cdot\,[\,mr + \rho\frac{r^2}{2} + \rho\frac{r_o^{\,2}}{2} + (m + \rho r)\,r_o\,]\right\}$$

which is of the form:

$$\boxed{\ddot{\phi} + 2\beta\dot{\phi} + \Omega^2\phi = F(t)} \qquad (2\text{-}4)$$

This is an equation of forced oscillation, in which Ω^2 is the square of the natural frequency, 2β is the damping term, and F(t) is the term forcing the oscillation.

For coupled oscillation mode, $\ddot{\theta}$ elimination gives the following ϕ - equation of motion:

$$\ddot{\phi}\left\{mr^2+\rho\frac{r^3}{3}-\frac{mr\,(r+r_o)}{I_T}\left[mr^2+\rho\frac{r^3}{3}+(mr+\rho\frac{r^2}{2})\,r_o\cos\phi\right]\right\}$$

$$+\dot{\phi}\left\{2\dot{r}\,(mr+\rho\frac{r^2}{2})+k-\frac{2\,\dot{r}m\,(r+r_o)}{I_T}\left[mr^2+\rho\frac{r^3}{3}+\right.\right.$$

$$\left.\left.(mr+\rho\frac{r^2}{2})\,r_o\cos\phi\right]\right\}+\phi\left\{\omega^2\,(mr+\rho\frac{r^2}{2})\,r_o+s\right\}$$

$$=-2\omega\dot{r}\left\{mr+\rho\frac{r^2}{2}-\frac{1}{I_T}\left[mr^2+\rho\frac{r^3}{3}+(mr+\rho\frac{r^2}{2})\,r_o\right]\cdot\left[mr+\right.\right.$$

$$\left.\left.\rho\frac{r^2}{2}+\rho\frac{r_o^2}{2}+(m+\rho r)\,r_o\right]\right\}$$

which is, like the uncoupled case, of the form:

$$\boxed{\ddot{\phi}+2\beta\dot{\phi}+\Omega^2\phi=F\,(t)}$$

In the limit of heavy hub, $I_T\to\infty$, no distinction can be made on the coupled from the uncoupled case, because $1/I_T$ terms go to zero. The forced oscillation equation will be solved analytically later, by Laplace's transform method.

The terms Ω, β, and F are listed below, including and neglecting wire mass:-

Uncoupled case:

$$\Omega^2=\frac{\omega^2(mr+\rho\frac{r^2}{2})\,r_o+s}{mr^2+\rho\frac{r^3}{3}}$$

$$2\beta = \begin{cases} [\,2\dot{r}\,(rm + \rho\, r^2/2\,) + k]\;/\;(mr^2 + \rho\,\frac{r^3}{3}) & t < \tau \\ k/\,(mr^2 + \rho\,\frac{r^3}{3}\,) & t > \tau \end{cases}$$

$$F(t) = \frac{-2\omega\dot{r}}{(mr^2 + \rho\,\frac{r^3}{3}\,)} \left\{ mr + \rho\,\frac{r^2}{2} - \frac{1}{I_T}\,[\,mr^2 + \rho\,\frac{r^3}{3} + (mr + \rho\,\frac{r^2}{2}\,)\,r_o\,]\;[\,m(r + r_o\,) + \rho\,(r^2 + r_o{}^2\,)\,/2 + \rho\, r\, r_o\,] \right\}$$

Coupled case

$$\Omega^2 = (\,\omega^2\,(\,mr + \rho\,\frac{r^2}{2}\,)\,r_o + s\,)\,/\,A$$

$$2\beta = \begin{cases} [\,2\,\dot{r}\,(\,mr + \rho\,\frac{r^2}{2}\,) - 2\,\dot{r}\,m\,(r + r_o\,)\,(\,mr^2 + \rho r^3/3 + (mr + \rho\, r^2/2\,) \\ \quad r_o)\,/\,I_T\,]\;/\;A + k/A & t < \tau \\ k/A & t > \tau \end{cases}$$

$$F(t) = \frac{-2\,\omega\,\dot{r}}{A} \left\{ m\,r + \rho\,\frac{r^2}{2} - \frac{1}{I_T}\,[\,mr^2 + \rho\,\frac{r^3}{3} + (\,mr + \rho\,\frac{r^2}{2})\,r_o\,]\cdot \right.$$
$$\left. \cdot\,[\,m(r + r_o) + \rho\,(r^2 + r_o{}^2\,)/2 + \rho\, r\, r_o\,] \right\}$$

$$A = mr^2 + \rho\,\frac{r^3}{3} - \frac{mr\,(r + r_o)}{I_T} \left[mr^2 + \rho\,\frac{r^3}{3} + (mr + \rho\,\frac{r^2}{2}\,)\,r_o \right]$$

$$\lim_{\rho \to 0} A = mr^2\, I_o/I_T$$

In the limit $\rho \to 0$, the terms β, Ω, and F are listed as following:

Uncoupled Case (Limit $\rho \to 0$)

$$\Omega = (\omega_o^2 r_o/r + s/mr^2)^{1/2}$$

$$\beta = \begin{cases} \dfrac{\dot{r}}{r} + \dfrac{k}{2mr^2} & t < \tau \\ \dfrac{k}{2mr^2} & t > \tau \end{cases}$$

$$F(t) = -\frac{2\,\omega_o \dot{r}}{mr^2}\left[mr - \frac{m^2 r\,(r + r_o)^2}{I_T} \right] \qquad (2\text{-}5)$$

$$= -\frac{2\,\omega_o\,\dot{r}\,I_o}{r\,I_T}$$

Coupled Case (Limit $\rho \to 0$)

$$\Omega = \left[\left(\omega_o^2 \frac{r_o}{r} + \frac{s}{mr^2}\right) \frac{I_T}{I_o} \right]^{1/2}$$

$$\beta = \begin{cases} \dfrac{\dot{r}}{r} + \dfrac{k}{2mr^2}\dfrac{I_T}{I_o} & t < \tau \\ \dfrac{k\,I_T}{2mr^2\,I_o} & t > \tau \end{cases}$$

$$F(t) = -\frac{2\,\omega_o\,\dot{r}}{r}\left\{ \frac{I_T}{I_o} - \frac{m\,(r + r_o)^2}{I_o} \right\} \qquad (2\text{-}6)$$

$$= -\frac{2\,\omega_o\,\dot{r}}{r}$$

2.3 Laplace Transforms and Transient Response

For short duration τ of deployment/retraction, such that $\dot{r}\tau \ll r$, the boom lengths remain approximately constant throughout the period τ. Thus, it is observed that the force term F(t) is of the form

$$F(t) \simeq \dot{r} \times \text{constant} \qquad 0 < t < \tau \qquad (2\text{-}7)$$

where $\dot{r}$ is a known constant. Transient response function ϕ (r, t) can be derived analytically by Laplace transform method. The scheme is to find the response ϕ (r, t) to the force $F(t) \cdot \theta(t)$ turned on at t=0, and then solve the force free oscillation equation using $\phi(r, \tau)$ as the initial conditions.

Using Laplace transforms, the forced oscillation equation becomes:

$$\mathcal{L}[\ddot{\phi} + 2\beta\dot{\phi} + \Omega^2 \phi] = \mathcal{L}[F.\theta(t)] \qquad (2\text{-}8)$$

$$\therefore \quad (s^2 + 2\beta s + \Omega^2)\,\phi(s) = \frac{F}{s}$$

$$\text{i.e.} \quad \phi(s) = \frac{F}{s(s^2 + 2\beta s + \Omega^2)}$$

By partial fractions, ϕ (s) can be written as:

$$\phi(s) = \frac{F}{\Omega^2}\left[\frac{1}{s} - \frac{s+\beta}{(s+\beta)^2 + \Omega_o^2} + \frac{\beta}{(s+\beta)^2 + \Omega_o^2}\right]$$

where $\Omega_o^2 = \Omega^2 - \beta^2$

Response function ϕ of the wire booms is given by

$$\phi(t) = \mathcal{L}^{-1}[\phi(s)]$$

Thus, the transient response function is found to be

$$\phi(t) = \frac{F}{\Omega^2}\left\{\theta(t) - e^{-\beta t}\left(\cos\Omega_o t + \frac{\beta}{\Omega_o} t \sin\Omega_o t\right)\right\} \qquad 0<t<\tau \qquad (2\text{-}9)$$

$$\text{where} \quad \theta(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

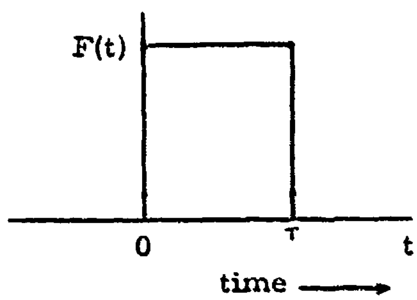
At $t = \tau$, in particular, the function $\phi(\tau)$ is

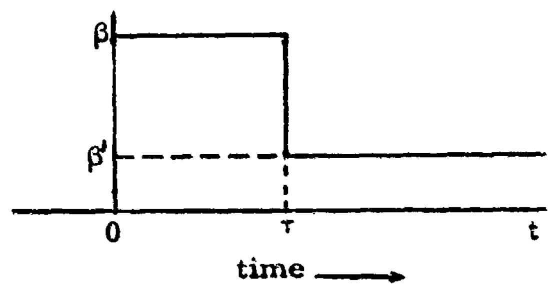
$$\phi(\tau) = \frac{F}{\Omega^2}\left\{ 1 - e^{\beta\tau}\left(\cos \Omega_o \tau + \frac{\beta}{\Omega_o} \sin \Omega_o \tau \right) \right\}$$

The next step is to solve the force free oscillation equation using $\phi(\tau)$ as the initial condition. Let us denote $\beta = \beta'$ during free oscillation,

i.e.

$$\beta = \begin{cases} \beta & \text{during force application} \quad (\tau > t > 0) \\ \beta' & \text{after force application} \quad (t > \tau) \end{cases}$$





For $t > \tau$, the equation of motion is of the form:

$$\ddot{\phi} + 2\beta'\dot{\phi} + \Omega^2\phi = 0$$

with known boundary conditions $\phi(\tau)$ and $\dot{\phi}(\tau)$.

For simplicity in symbol manipulation, let $\tau = 0$ from here on, until later when we restore $t = t + \tau$, $\tau \neq 0$. By partial integration,

$$\mathcal{L}[\dot{\phi}(t)] = \int_0^\infty dt e^{-st}\dot{\phi}(t) = \left[e^{-st}\phi(t) \right]_0^\infty + s\int_0^\infty dt e^{-st}\phi(t)$$

$$= -\phi(0) + s\mathcal{L}[\phi(t)]$$

$$\mathcal{L}[\ddot{\phi}(t)] = \int_0^\infty dt e^{-st}\ddot{\phi}(t) = \left[e^{-st}\dot{\phi}(t) \right]_0^\infty + s\int_0^\infty dt e^{-st}\dot{\phi}(t)$$

$$= -\dot{\phi}(0) - s\phi(0) + s^2\mathcal{L}[\phi(t)]$$

Therefore, the equation of motion is transformed as follows:

$$(s^2 + 2\beta' s + \Omega^2)\ \phi(s) = s\,\phi(0) + \dot{\phi}(0) + 2\beta'\ \phi(0)$$

or, $$\phi(s) = \phi(0)\frac{s + \beta'}{(s+\beta')^2 + \Omega_o'^2} + [\beta\,\phi(0) + \dot{\phi}(0)]\frac{1}{(s+\beta')^2 + \Omega_o'^2}$$

where $\Omega_o'^2 = \Omega^2 - \beta'^2$

Thus,

$$\phi(t) = \phi(0)\,e^{-\beta' t}\cos\Omega_o' t + \left[\frac{\beta'\phi(0) + \dot{\phi}(0)}{\Omega_o'}\right] e^{-\beta' t}\sin\Omega_o' t$$

Restoring t to $t + \tau$, $\tau \neq 0$ by shifting origin: we find

$$\phi(t) = \phi(\tau)\,e^{-\beta'(t-\tau)}\cos[\Omega_o'(t-\tau)] + \left\{\frac{\beta'\phi(\tau) + \dot{\phi}(\tau)}{\Omega_o'}\right\} e^{-\beta'(t-\tau)}\,.$$

$$\cdot\ \sin[\Omega_o'(t-\tau)] \qquad t > \tau \tag{2-10}$$

where $\phi(\tau)$ and $\dot{\phi}(\tau)$ are known boundary conditions:

$$\phi(\tau) = \frac{F}{\Omega^2}\left\{1 - e^{-\beta\tau}\left[\cos\Omega_o\tau + \frac{\beta}{\Omega_o}\sin\Omega_o\tau\right]\right\} \tag{2-11}$$

$$\dot{\phi}(\tau) = \frac{F\Omega_o}{\Omega^2}\left\{\left(\frac{\beta}{\Omega_o}\right)^2 + 1\right\} e^{-\beta\tau}\sin\Omega_o\tau$$

$$= \frac{F}{\Omega_o}\,e^{-\beta\tau}\sin\Omega_o\tau \tag{2-12}$$

where $\Omega_o^2 = \Omega^2 - \beta^2$

2.4 Physical Insights gained from these Results

With the use of a simple model in this introductory chapter, we have been able to derive some salient physical features of the satellite system concerned. The important points are listed below:

1) Retraction is less stable, because $\dot{r}$ is negative reversing the sign of the damping term β. [Equs (2-5), (2-6)].

2) Damping is prominent when boom is short, because β is proportional to $1/r^2$. [Equ. (2-5), (2-6)].

3) Force F(t) is prominent when boom is short, because F(t) is proportional to $1/r$. [Equ. (2-5), (2-6)].

4) If k is very large, no oscillation occurs, because then ϕ (t) is an exponentially decaying function. For, let $\beta > \Omega$, so that $\Omega_o = i\alpha$, where α is real and $\alpha < \beta$ because $\Omega_o = i\sqrt{\beta^2 - \Omega^2} = i\alpha$, and then for $\tau > t > 0$, we have; from Equ. (2-9), :

$$\phi(t) = \frac{F}{\Omega^2}\left\{1 - e^{-\beta t}\left[\frac{e^{-\alpha t} + e^{\alpha t}}{2} + \frac{\beta}{i\alpha}\,\frac{e^{-\alpha t} - e^{\alpha t}}{2i}\right]\right\}$$

$$= \frac{F}{\Omega^2}\left\{1 - \frac{e^{-(\beta-\alpha)t}}{2\alpha}\left[(\alpha-\beta)e^{-2\alpha t} + \alpha + \beta\right]\right\}$$

which decays exponentially.

5) The Amplitude of Oscillation after Deployment/Retraction Period τ depends on when the deployment/retraction stops, because ϕ $(t > \tau)$ is a function of $\phi(\tau)$ and $\dot{\phi}(\tau)$, [Equ. (2-10)].

CHAPTER 3

LAGRANGIAN DYNAMICS OF COUPLED HUB-BOOM INCLUDING TRANSLATION

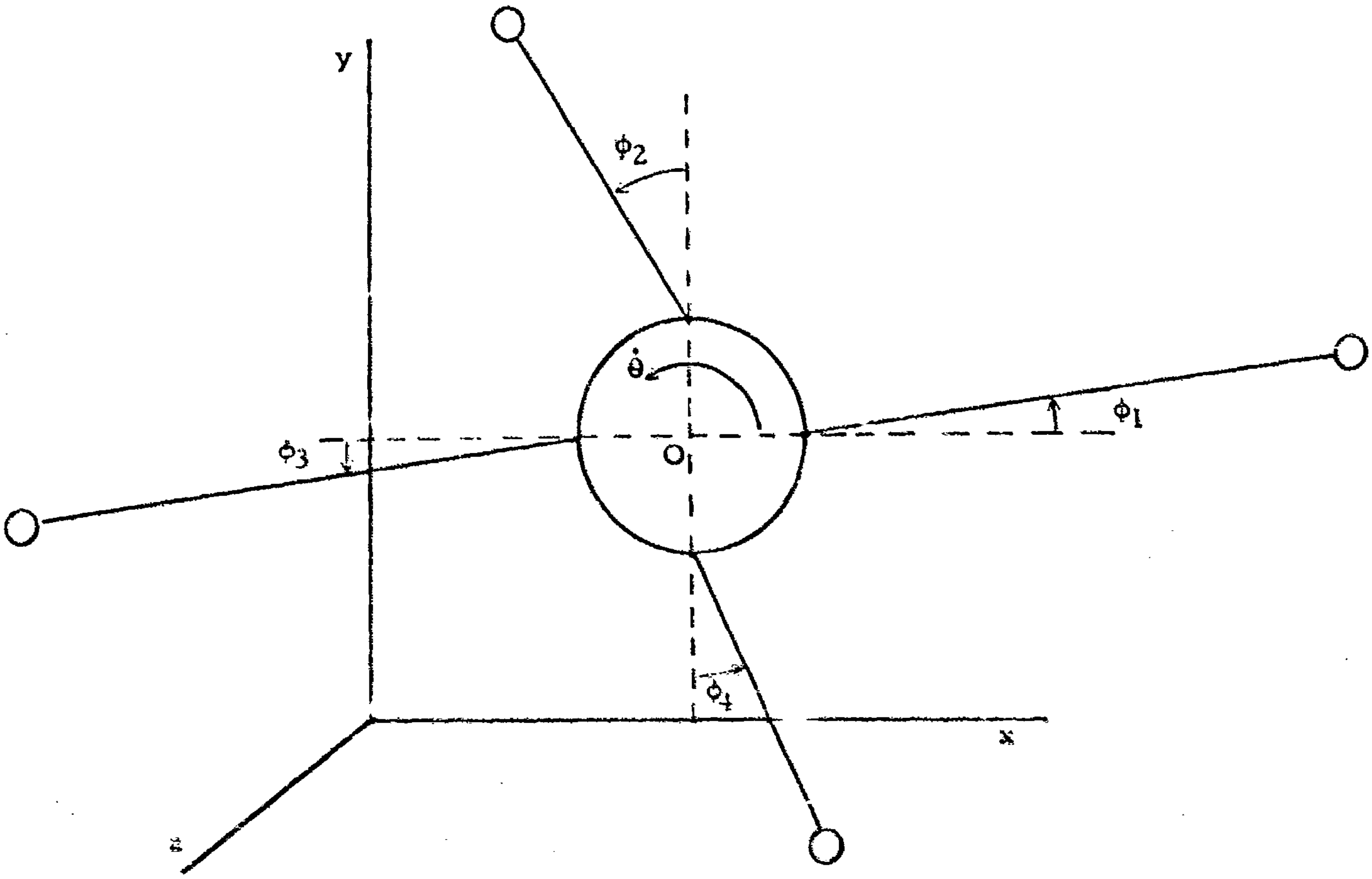


Figure 2. Spinning Satellite-Wire Boom System

3.1 Notations:

$\underline{r}$ = (x, y, z) = Rectangular coordinates in earth's inertial frame.

$\underline{R}$ = (X, Y, Z) = Position of center of hub w.r.t initial position of center of hub (t = 0) as measured in earth's inertial coordinates

Note: For negligible or completely independent out-of-plane motions, z and Z can be omitted in the in-plane dynamics

O = Center of hub

θ = Angle subtended by the straight line defined by O and the exit point of the first boom w.r.t x-axis in earth's inertial frame. At t = O, $\theta\big|_{t=O}$ = O initially, and $\dot{\theta}\big|_{t=O}$ = initial angular velocity of the hub, $(\omega = \dot{\theta})$.

(r_i, ϕ_i) = Polar coordinates of boom-tip mass (i - th) as measured in corotating coordinates. The i's are chosen so that initially the first boom subtends an angle θ = O, w.r.t earth's x-axis. $\phi_i(t)$ is the angular displacement of boom i at time t.

M = Mass of hub

m = Mass of a tip mass

r_o = Radius of hub

I_o = Moment of inertia of hub

I_T = Total moment of inertia of system

s = Spring constant of boom wire

k = Air drag constant, (for air drag on boom).

ρ = Mass density of wire

L = Lagrangian of total system

L_o = Lagrangian of hub

L_i = Lagrangian of i - th boom with tip mass (due to pure rotation only)

L_i' = Lagrangian of i - th boom with tip mass (due to cross effect of rotation and translation)

$A_i = mr_i^2 + \rho r_i^3/3$

$B_i = mr_i + \rho r_i^2/2$

$D_i = m + \rho r_i$

$\delta_i = (i - 1)\frac{\pi}{2} \quad (i = 1, \ldots .4)$

$\Phi_i = \phi_i + \theta + \delta i$

$\Theta_i = \theta + \delta_i$

$\underline{\dot{r}}$ = Velocity relative to inertial axes

$\underline{\dot{r}'}$ = Velocity relative to rotating axes

$\underline{\dot{R}}$ = Translational velocity of moving axes origin w.r.t. inertial axes

The vector equation relating to motion in a rotating frame and that in an inertial frame is

$$\underline{\dot{r}} = \underline{\dot{R}} + \underline{\dot{r}'} + \underline{\omega} \times \underline{r}' \qquad (3\text{-}1)$$

where $\underline{\dot{r}}$ is the velocity in an inertial frame,

$\underline{\dot{R}}$ is the translational velocity of the origin of the rotating frame,

$\underline{\dot{r}'}$ is the velocity w.r.t. rotating axes,

ω is the angular velocity of the rotating axes.

Note: All four variables above are functions of time t.

In components,

$$\dot{x} = \dot{X} + \dot{x}' - \omega y' \qquad (3\text{-}2)$$
$$\dot{y} = \dot{Y} + \dot{y}' + \omega x'$$

Note: These equations (3-1, 2) can be derived by considering infinitesimal displacements (and therefore derivatives) in the inertial frame.

It is convenient to change (x_i, y_i) to polar coordinates (r_i, ϕ_i), but leaving (X, Y) unchanged in rectangular coordinates.

$$x_i = r_i \cos(\phi_i + \delta_i + \theta) + r_o \cos(\delta_i + \theta) \qquad (3\text{-}3)$$
$$y_i = r_i \sin(\phi_i + \delta_i + \theta) + r_o \sin(\delta_i + \theta)$$

where $\delta_i = (i - 1)\frac{\pi}{2}$ $\quad$ (i = 1, ..., 4)

Let $\Phi_i = \phi_i + \delta_i + \theta$, $\dot{\Phi}_i = (\dot{\phi}_i + \dot{\theta})$, $\ddot{\Phi}_i = \ddot{\phi}_i + \ddot{\theta}$

$\Theta_i = \delta_i + \theta$, $\dot{\Theta}_i = \dot{\theta}$, $\ddot{\Theta}_i = \ddot{\theta}$

$$x_i = r_i \cos \Phi_i + r_o \cos \Theta_i$$

$$y_i = r_i \sin \Phi_i + r_o \sin \Theta_i$$

$$\dot{x}_i = \dot{r}_i \cos \Phi_i - r_i \sin \Phi_i \dot{\Phi}_i - r_o \sin \Theta_i \dot{\theta}$$

$$\dot{y}_i = \dot{r}_i \sin \Phi_i + r_i \cos \Phi_i \dot{\Phi}_i + r_o \cos \Theta_i \dot{\theta}$$

$$\ddot{x}_i = -2 \dot{r}_i \sin \Phi_i \dot{\Phi}_i - r_i \cos \Phi_i \dot{\Phi}_i^2 - r_i \sin \ddot{\Phi}_i - r_o \cos \Theta_i \dot{\theta}^2 - r_o \sin \Theta_i \ddot{\theta}$$

$$\ddot{y}_i = 2 \dot{r}_i \cos \Phi_i \dot{\Phi}_i - r_i \sin \Phi_i \dot{\Phi}_i^2 + r_i \cos \Phi_i \ddot{\Phi}_i - r_o \sin \Theta_i \dot{\theta}^2 + r_o \cos \Theta_i \ddot{\theta}$$

The time changes of x_i, y_i as given above are those observed in the earth's inertial frame.

3.2 Kinetic Energy of a Tip Mass

Kinetic energy (K. E.) of the i-th tip mass is

$$K.E. = 1/2\, m \left[(\dot{X} + \dot{x}_i)^2 + (\dot{Y} + \dot{y}_i)^2 \right]$$

$$= 1/2\, m \,(\dot{X}^2 + \dot{Y}^2) + 1/2\, m \,(\dot{x}_i^{\,2} + \dot{y}_i^{\,2}) +$$

$$1/2\, m \,(2\dot{X}\dot{x}_i + 2\dot{Y}\dot{y}_i)$$

where the first term is the pure translation term $K.E._{trans}$, the second term is the pure rotation term $K.E._{rot}$, and the third term the cross term $K.E._{cross}$.

$$K.E._{trans} \quad 1/2\, m \,(\dot{X}^2 + \dot{Y}^2)$$

$$K.E._{rot} = 1/2\, m \,(\dot{x}_i^{\,2} + \dot{y}_i^{\,2})$$

$$= 1/2\, m \left[\dot{r}_i^{\,2} + r_o^{\,2}\dot{\theta}^2 + r_i^{\,2}(\dot{\theta} + \dot{\phi}_i)^2 + \right.$$

$$\left. 2r_o\dot{\theta}\,(r_i[\dot{\theta} + \dot{\phi}_i]\cos\phi_i + \dot{r}_i\sin\phi_i) \right]$$

which is <u>independent</u> of δ_i

$$K.E._{cross} = m \left[\dot{\theta}([r_i\cos\Phi_i + r_o\cos\Theta_i]\dot{Y} - [r_i\sin\Phi_i + r_o\sin\Theta_i]\dot{X}) + \right.$$

$$(\dot{r}_i\cos\Phi_i - r_i\sin\Phi_i\,\phi_i)\,\dot{X} +$$

$$\left. (\dot{r}_i\sin\Phi_i + r_i\cos\Phi_i\,\dot{\phi}_i)\,\dot{Y} \right]$$

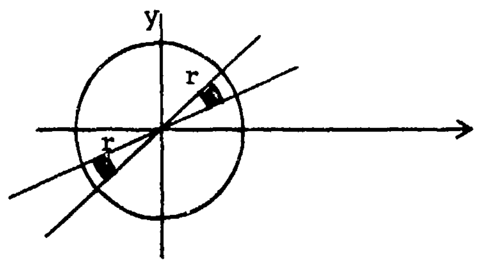
For <u>boom wire</u>, one replaces

$$mr^n \rightarrow \int_o^r dr\rho r^n = \rho \frac{r^{(n+1)}}{(n+1)} \qquad (3\text{-}4)$$

To show that K. E. of hub = K. E.$_{\text{rotation}}$ + K. E.$_{\text{translation}}$ without any cross term:

$$K.E._{cross} = 0$$

Proof (1): Take x axis to be parallel to the translation vector



Take 2 elements opposite and equidistant from center of of hub, their velocities are

$$(\dot{x}_\omega - \dot{x}_t, \ \dot{y}_\omega)$$

and $(\dot{x}_\omega + \dot{x}_t, \ -\dot{y}_\omega)$ respectively.

In K.E., one needs $\sum_i (\text{velocity})_i^2$ for all elements. Since cross terms cancel in $(\text{velocity})_i^2$ for all opposite pairs,

$$\therefore \ K.E._{total} = K.E._{rot} + K.E._{trans} \qquad Q.E.D.$$

Proof (2): Let v = velocity of an element of hub

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

where
$$\dot{x} = \dot{X} - \omega y$$
$$\dot{y} = \dot{Y} + \omega x$$
$$\dot{z} = 0$$

$$v^2 = \omega^2 (x^2 + y^2) + (\dot{X}^2 + \dot{Y}^2) + 2\omega (x\dot{Y} - y\dot{X})$$

$$K.E._{hub} = K.E._{rot.} + K.E._{trans.} + K.E._{cross}$$

where
$$K.E._{cross} = 1/2\rho \int dr r^2 \, d\theta \, d\phi \, \sin\theta \ 2\omega \ (x\dot{Y} - y\dot{X})$$
$$= \rho\omega \int dr r^2 \, d\theta \, d\phi \ \sin\theta \ (r \sin\theta \cos\phi \ \dot{Y} - r \sin\theta \sin\phi \dot{X})$$
$$= \rho\omega \int_0^{r_o} dr r^3 \int_{-\pi}^{\pi} d\theta \sin^2\theta \int_0^{2\pi} d\phi (\cos\phi \ \dot{Y} - \sin\phi \ \dot{X})$$
$$= 0 \quad Q.E.D.$$

3.3 Total Lagrangian

The Lagrangian L of the entire system can therefore be written down as follows:

$$L = L_o(\dot\theta, \dot X, \dot Y) + \sum_{i=1}^{4} L_i(\dot\theta, r_i, \dot r_i, \phi_i, \dot\phi_i) + \sum_{i=1}^{4} L_i'(\dot\theta, r_i, \dot r_i, \phi_i, \dot\phi_i, \delta_i, \dot X, \dot Y) \qquad (3\text{-}5)$$

where L_o is the Lagrangian of the hub, L_i is the Lagrangian of the boom-tip mass due to pure rotation, and L_i' is the Lagrangian of the boom-tip mass system due to cross effect of rotation-translation.

$$L_o = 1/2\, I_o\, \dot\theta^2 + 1/2\, M (\dot X^2 + \dot Y^2)$$

$$\begin{aligned} L_i = {} & 1/2\,(m + \rho r_i)\dot r_i^{\,2} + 1/2\,(m r_i^2 + \rho \frac{r_i^2}{3})\,\dot\phi_i^2 + \\ & \dot\theta[\,(m r_i^2 + \rho\frac{r_i^2}{3})\,\dot\phi_i + (m + \rho r_i)\, r_o\, \dot r_i \sin\varphi_i + \\ & (m r_i + \rho\frac{r_i^2}{2})\, r_o \cos\phi_i\, \dot\phi_i\,] - 1/2\, s\phi_i^2 + \\ & 1/2\,\dot\theta^2\,[\,(m r_i^2 + \rho\frac{r_i^3}{3}) + (m + \rho r_i)\, r_o^2 + 2(m r_i + \rho\frac{r_i^2}{2}) r_o \cos\phi_i\,] \end{aligned}$$

$$\begin{aligned} L_i' = {} & 1/2\,(m + \rho r_i)(\dot X^2 + \dot Y^2) + \dot\theta[\,(m + \rho r_i)(\, r_o\cos\Theta_i\, \dot Y - r_o \sin\Theta_i\, \dot X) + \\ & (m r_i + \rho\frac{r_i^2}{2})(\,\dot Y\cos\Phi_i - \dot X\sin\Phi_i\,)\,] + \\ & (m + \rho r_i)\,\dot r_i\,(\dot X\cos\Phi_i + \dot Y\sin\Phi_i\,) + (m r_i + \rho\frac{r_i^2}{2})\,\dot\phi_i\,(\,-\dot X\sin\Phi_i + \dot Y\cos\Phi_i) \end{aligned}$$

The Lagrangian equations of motion are:

θ-equ: $$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta} - \frac{\partial L}{\partial\theta} = -k_\theta\dot\theta$$

ϕ_i-equ: $$\frac{d}{dt}\frac{\partial L}{\partial\dot\phi_i} - \frac{\partial L}{\partial\phi} = -k_\phi\dot\phi \qquad (i = 1, \ldots, 4)$$

r_i-equ: $$\frac{d}{dt}\frac{\partial L}{\partial\dot r_i} - \frac{\partial L}{\partial r_i} = -T_i \qquad (i = 1, \ldots, 4)$$

X -equ: $\frac{d}{dt}\frac{\partial L}{\partial \dot{X}} - \frac{\partial L}{\partial X} = 0$ (if there is no air drag due to translation)

Y -equ: $\frac{d}{dt}\frac{\partial L}{\partial \dot{Y}} - \frac{\partial L}{\partial Y} = 0$ (")

The X and Y equs can also be obtained alternatively by considering conservations of linear momenta in X and Y directions respectively (see Appendix A).

The θ-equ. can also be obtained alternatively by considering conservation of angular momentum of the entire system.

At high altitudes (ionospheric for satellite 1975), air drag is negligible (see Chapter 7), so that k_{ϕ} is mainly due to hinge friction and k_{θ} is negligible. From here on, we use k in place of k_{ϕ} and k_{θ} is neglected.

3.4 θ - equation of Motion

$$\frac{\partial L}{\partial \theta} = -\sum_{i=1}^{4} [\,\dot\theta\,(D_i r_o [\,\dot Y \sin\Theta_i + \dot X \cos\Theta_i] + B_i [\,\dot Y \sin\Phi_i + \dot X \cos\Phi_i]\,) +$$
$$D_i \dot r_i (\dot X \sin\Phi_i - \dot Y \cos\Phi_i) + B_i \dot\phi_i (\dot X \cos\Phi_i + \dot Y \sin\Phi_i)]$$

$$\frac{\partial L}{\partial \dot\theta} = I_o \dot\theta + \sum_{i=1}^{4} [\,A_i (\dot\theta + \dot\phi_i) + B_i r_o \cos\phi_i (2\dot\theta + \dot\phi_i) + D_i r_o (r_o \dot\theta + \dot r_i \sin\phi_i)\,] +$$
$$\sum_{i=1}^{4} [B_i (\dot Y \cos\Phi_i - \dot X \sin\Phi_i) + D_i r_o (\dot Y \cos\Theta_i - \dot X \sin\Theta_i)\,]$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = I_o \ddot\theta + \sum_{i=1}^{4} [\,\dot A_i (\dot\theta + \dot\phi_i) + A_i (\ddot\theta + \ddot\phi_i) + \dot B_i r_o \cos\phi_i (2\dot\theta + \dot\phi_i) +$$
$$B_i r_o \cos\phi_i (2\ddot\theta + \ddot\phi) - B_i r_o \sin\phi_i\, \dot\phi_i (2\dot\theta + \dot\phi_i) +$$
$$\dot D_i r_o (r_o \dot\theta + \dot r_i \sin\phi_i) + D_i r_o (r_o \ddot\theta + \dot r_i \cos\phi_i \dot\phi_i) +$$
$$\dot B_i (\dot Y \cos\Phi_i - \dot X \sin\Phi_i) +$$
$$B_i (\ddot Y \cos\Phi_i - \dot Y \sin\Phi_i [\,\dot\theta + \dot\phi_i] - \ddot X \sin\Phi_i - X \cos\Phi_i [\dot\theta + \dot\phi_i]\,) +$$
$$\dot D_i r_o (\dot Y \cos\Theta_i - \dot X \sin\Theta_i) +$$
$$D_i r_o (\ddot Y \cos\Theta_i - \ddot X \sin\Theta_i - \dot Y \sin\Theta_i \dot\theta - \dot X \cos\Theta_i \dot\theta)\,] \qquad (3\text{-}6)$$

Since total moment of inertia of the system is the sum of moment of inertia of the hub plus those of the booms with tip mass, therefore the total moment of inertia is:

$$I_T = I_o + \sum_{i=1}^{4} [\,(mr_i^2 + \rho\frac{r_i^3}{3}) + (m + \rho r_i)\, r_o^2 +$$
$$2(\,mr_i + \rho\frac{r_i^2}{2})\, r_o \cos\phi_i\,] \qquad (3\text{-}7)$$

ie. $$I_T = I_o + \sum_{i=1}^{4} (\,A_i + 2B_i r_o \cos\phi_i + D_i r_o^2)$$

This expression is now put into the R.H.S. of equation (3-6) given on the previous page. The θ-equation of motion is then written as follows:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} - \frac{\partial L}{\partial \theta} = I_T \ddot\theta + \sum_{i=1}^{4} \Big\{ 2 (\dot\theta + \dot\phi_i)(B_i \dot r_i + D_i r_o \cos\phi_i \dot r_i - B_i r_o \sin\phi_i \dot\phi_i) + \dot\phi_i B_i r_o \sin\phi_i \dot\phi_i + \ddot\phi_i (A_i + B_i r_o \cos\phi_i) + \rho r_o \dot r_i (r_o \theta + \dot r_i \sin\phi_i) - B_i (\ddot X \sin \Phi_i - \ddot Y \cos \Phi_i) + D_i r_o (\ddot Y \cos\Theta_i - \ddot X \sin\Theta_i) + \rho \dot r_i r_o (\dot Y \cos\Theta_i - \dot X \sin\Theta_i) \Big\} \qquad (3-8)$$

$$= \begin{cases} 0 & \text{if there is no air drag due to rotation of hub} \\ -k_\theta \dot\theta & \text{if there is air drag.} \end{cases}$$

In the limiting case, let $\dot X = \dot Y = 0$, one obtains the "no-translation" version of θ-equation.

3.5 ϕ_i - equation of motion

$$\frac{\partial L}{\partial \phi_i} = r_o \dot\theta \left(D_i \dot r_i \cos\phi_i - B_i \sin\phi_i \dot\phi_i \right) - r_o \dot\theta^2 B_i \sin\phi_i -$$

$$s\phi_i - \left[B_i \dot Y (\theta + \phi_i) + D_i \dot r_i \dot X \right] \sin\phi_i +$$

$$\left[D_i \dot r_i \dot Y - B_i \dot X (\dot\theta + \dot\phi_i) \right] \cos\phi_i$$

$$= r_o \dot\theta \left(D_i \dot r_i \cos\phi_i - B_i \sin\phi_i \dot\phi_i - B_i \sin\phi_i \dot\theta \right) - s\phi_i -$$

$$(\dot\theta + \dot\phi_i) B_i \left(\dot X \cos\phi_i + \dot Y \sin\phi_i \right) + D_i \dot r_i \left(\dot Y \cos\phi_i - \dot X \sin\phi_i \right)$$

$$\frac{\partial L}{\partial \dot\phi_i} = \dot\phi_i A_i + \dot\theta (A_i + B_i r_o \cos\phi_i) - B_i (\dot X \sin\phi_i - \dot Y \cos\phi_i)$$

$$= A_i (\dot\theta + \dot\phi_i) + B_i \left[\dot\theta r_o \cos\phi_i - (\dot X \sin\phi_i - \dot Y \cos\phi_i) \right]$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\phi_i} = \dot A_i (\dot\theta + \dot\phi_i) + A_i (\ddot\theta + \ddot\phi_i) + \dot B_i \dot\theta r_o \cos\phi_i +$$

$$B_i (\ddot\theta r_o \cos\phi_i - \dot\theta r_o \sin\phi_i \dot\phi_i) + \dot B_i (\dot Y \cos\phi_i - \dot X \sin\phi_i) -$$

$$B_i \left[(\dot X \cos\phi_i + \dot Y \sin\phi_i)(\dot\theta + \dot\phi_i) + (\ddot X \sin\phi_i - \ddot Y \cos\phi_i) \right]$$

$$= \ddot\theta (A_i + B_i r_o \cos\phi_i) + A_i \ddot\phi_i + 2B_i \dot r_i (\dot\theta + \dot\phi_i) +$$

$$D_i \dot r_i \dot\theta r_o \cos\phi_i - B_i r_o \dot\theta \sin\phi_i \dot\phi_i + D_i \dot r_i (\dot Y \cos\phi_i - \dot X \sin\phi_i) -$$

$$B_i \left[(\dot X \cos\phi_i + \dot Y \sin\phi_i)(\dot\theta + \dot\phi_i) + (\ddot X \sin\phi_i - \ddot Y \cos\phi_i) \right]$$

Collecting the results, we therefore have the ϕ_i - equation of Motion:

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi_i} &= A_i \ddot{\phi}_i + 2B_i \dot{r}_i (\dot{\theta} + \dot{\phi}_i) + \ddot{\theta}(A_i + B_i r_o \cos\phi_i) + \\
&\quad s\phi_i + \dot{\theta}^2 B_i r_o \sin\phi_i + B_i(\dot{\theta} + \dot{\phi}_i)(\dot{X}\cos\phi_i + \dot{Y}\sin\phi_i) \\
&\quad + B_i\left[-(\dot{\theta} + \dot{\phi}_i)(\dot{X}\cos\phi_i + \dot{Y}\sin\phi_i) + \ddot{Y}\cos\phi_i - \ddot{X}\sin\phi_i\right] \\
&= A_i \ddot{\phi}_i + 2B_i \dot{r}_i (\dot{\theta} + \dot{\phi}_i) + s\phi_i + \dot{\theta}^2 B_i r_o \sin\phi_i + \\
&\quad \ddot{\theta}(A_i + B_i r_o \cos\phi_i) + B_i(\ddot{Y}\cos\phi_i - \ddot{X}\sin\phi_i) \\
&= -k\dot{\phi}_i \quad (i = 1, \ldots, 4) \qquad (3\text{-}9)
\end{aligned}
$$

In the limiting case of "no-translation", one lets $\dot{X}$, $\dot{Y} = 0$ and obtains the ϕ_i - equation of motion.

3.6 X-equation of Motion

$$\frac{\partial L}{\partial X} = 0$$

$$\frac{\partial L}{\partial \dot{X}} = M\dot{X} + \sum_{i=1}^{4} \left[(m + \rho r_i)(\dot{X} + \dot{r}_i \cos\phi_i - \dot{\theta} r_o \sin\Theta_i) - \right.$$

$$\left. (m r_i + \rho r_i^2/2)(\dot{\theta}\sin\phi_i + \dot{\phi}_i \sin\phi_i) \right]$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{X}} = M\ddot{X} + \sum_{i=1}^{4} \left[\rho\, \dot{r}_i (\dot{X} + \dot{r}_i \cos\phi_i - r_o \dot{\theta} \sin\Theta_i) + (m + \rho r_i) \right.$$

$$(\ddot{X} - \dot{r}_i \sin\phi_i\, [\dot{\theta} + \dot{\phi}_i] - \ddot{\theta} r_o \sin\Theta_i -$$

$$r_o \cos\Theta_i\, \dot{\theta}^2) - (m r_i + \rho r_i^2/2)([\ddot{\theta} + \ddot{\phi}_i] \sin\phi_i + [\dot{\theta} + \dot{\phi}_i]$$

$$\left. \cos\phi_i\, [\dot{\theta} + \dot{\phi}_i]) - (m + \rho r_i)\dot{r}_i(\dot{\theta} + \dot{\phi}_i)\sin\phi_i \right]$$

The X - equation of Motion is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{X}} - \frac{\partial L}{\partial X} = M\ddot{X} + \sum_{i=1}^{4} \left[\rho \dot{r}_i (\dot{X} + \dot{r}_i \cos\phi_i - r_o \dot{\theta}\sin\Theta_i) + \right.$$

$$(m + \rho r_i)(\ddot{X} - 2\dot{r}_i [\dot{\theta} + \dot{\phi}_i] \sin\phi_i - [\ddot{\theta}\sin\Theta_i + \dot{\theta}^2 \cos\Theta_i] r_o)$$

$$\left. - (m r_i + \rho \frac{r_i^2}{2})([\ddot{\theta} + \ddot{\phi}_i]\sin\phi_i + [\dot{\theta} + \dot{\phi}_i]^2 \cos\phi_i) \right] \tag{3-10}$$

$= 0$ if there is negligible air drag due to translation

3.7 Y-equation of Motion

$$\frac{\partial L}{\partial Y} = 0$$

$$\frac{\partial L}{\partial \dot{Y}} = M\dot{Y} + \sum_{i=1}^{4} \Big[(m + \rho r_i)(\dot{Y} + \dot{r}_i \sin\phi_i + r_o \dot{\theta} \cos\Theta_i) + (m r_i + \rho \frac{r_i^2}{2})(\dot{\theta} + \dot{\phi}_i) \cos\phi_i \Big]$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{Y}} = M\ddot{Y} + \sum_{i=1}^{4} \Big\{ \rho \dot{r}_i (\dot{Y} + \dot{r}_i \sin\phi_i + r_o \dot{\theta} \cos\Theta_i) + (m + \rho r_i)(\ddot{Y} + \dot{r}_i \cos\phi_i [\dot{\theta} + \dot{\phi}_i] + r_o \ddot{\theta} \cos\Theta_i - r_o \dot{\theta}^2 \sin\Theta_i + \dot{r}_i [\dot{\theta} + \dot{\phi}_i] \cos\phi_i) + (m r_i + \rho \frac{r_i^2}{2})([\ddot{\theta} + \ddot{\phi}_i] \cos\phi_i - [\dot{\theta} + \dot{\phi}_i] \sin\phi_i [\dot{\theta} + \dot{\phi}_i]) \Big\}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{Y}} - \frac{\partial L}{\partial Y} = M\ddot{Y} + \sum_{i=1}^{4} \Big\{ \rho \dot{r}_i (\dot{Y} + \dot{r}_i \sin\phi_i + r_o \dot{\theta} \cos\Theta_i) + (m + \rho r_i)(\ddot{Y} + 2 \dot{r}_i [\dot{\theta} + \dot{\phi}_i] \cos\phi_i + r_o \ddot{\theta} \cos\Theta_i - r_o \dot{\theta}^2 \sin\Theta_i) + (m r_i + \rho \frac{r_i^2}{2})([\ddot{\theta} + \ddot{\phi}_i] \cos\phi_i - [\dot{\theta} + \dot{\phi}_i]^2 \sin\phi_i) \Big\} \qquad (3\text{-}11)$$

$= 0$ if there is negligible air drag due to translation

3.8 r_i - equation of Motion

$$\frac{\partial L}{\partial r_i} = 1/2\ \rho(\dot{r}_i^{\,2} + r^2\dot{\theta}^2) + B_i(\dot{\theta} + \dot{\phi}_i)^2 + \rho\, r_o \dot{r}_i \dot{\theta} \sin\phi_i +$$

$$D_i\, r_o \cos\phi_i\ \dot{\theta}(\dot{\theta} + \dot{\phi}_i) + \rho[\,1/2\,(\dot{X}^2 + \dot{Y}^2) + \dot{r}_i(\dot{X}\cos\phi_i + \dot{Y}\sin\phi_i)$$

$$+ \dot{\theta}(r_o \cos\Theta_i\, \dot{Y} - r_o \sin\Theta_i\, \dot{X})] + D_i(\dot{\theta} + \dot{\phi}_i)(\dot{Y}\cos\phi_i - \dot{X}\sin\phi_i)$$

$$\frac{\partial L}{\partial \dot{r}_i} = D_i(\dot{r}_i + r_o\dot{\theta}\sin\phi_i + \dot{X}\cos\phi_i + \dot{Y}\sin\phi_i)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}_i} = \dot{D}_i(\dot{r}_i + r_o\dot{\theta}\sin\phi_i) + D_i(r_o\ddot{\theta}\sin\phi_i + r_o\dot{\theta}\cos\phi_i\,\dot{\phi}_i) +$$

$$\dot{D}_i(\dot{X}\cos\phi_i + \dot{Y}\sin\phi_i) + D_i(\ddot{X}\cos\phi_i + \ddot{Y}\sin\phi_i - \dot{X}\sin\phi_i[\dot{\theta} + \dot{\phi}_i]$$

$$+ \dot{Y}\cos\phi_i[\dot{\theta} + \dot{\phi}_i])$$

r - equ $\therefore$ $$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}_i} - \frac{\partial L}{\partial r_i} = 1/2\ \rho(\dot{r}_i^{\,2} - r_o^{\,2}\dot{\theta}^2) + D_i\, r_o(\ddot{\theta}\sin\phi_i -$$

$$\dot{\theta}^2\cos\phi_i) - B_i(\dot{\theta} + \dot{\phi}_i)^2 - \frac{\rho}{2}\,[\,(\dot{X}^2 + \dot{Y}^2) +$$

$$2\dot{\theta}(r_o\cos\Theta_i\,\dot{Y} - r_o\sin\Theta_i\,\dot{X})\,] -$$

$$D_i(\ddot{X}\cos\phi_i + \ddot{Y}\sin\phi_i)$$

$$= -T_i \qquad (i = 1, \ldots, 4) \qquad (3\text{-}12)$$

CHAPTER 4

ANALYTIC SOLUTION - FOR INPLANE OSCILLATIONS

The Lagrangian equations of motion for inplane satellite dynamics form a set of seven coupled nonlinear differential equations, which can be solved numerically on a computer. To gain insights into the physical behavior of the system, analytic solutions are useful. Simplifications of the problem are necessary for the feasibility of analytic solutions. The simplifying assumptions are as follows:

1) No external damping
2) No deployment or retraction
3) Equal boom lengths
4) Harmonic approximation

In assumption (4), the boom angles ϕ_i are of the order $O(\epsilon)$, $\dot{\phi}_i$ are also of the order $O(\epsilon)$. Since $\dot{\theta} = \omega_0$ at equilibrium, $\dot{\theta}$ itself need not be small, but the deviation from ω_0 is. Thus, let $\dot{\theta} = \omega_0 + \dot{\theta}'$ where $\dot{\theta}'$ is of the order $O(\epsilon)$. The approximation scheme is "all terms of the order $O(\epsilon^3)$ or higher will be discarded in kinetic and potential energies, and, all terms of the order $O(\epsilon^2)$ or higher will be discarded in the equations of motion". The results arrived at by using equations of motion should be the same as those obtained from kinetic and potential energies [10, 12].

4.1 No-Translation Formulation

We first consider no-translation, and later translation will be included, so that the difference in results due to translational effect can be obtained.

In harmonic approximation, the matrix equation of motion (page 31) becomes

$$AX = B \tag{4-1}$$

where

$$[A] = \begin{pmatrix} a & 0 & 0 & 0 & b \\ 0 & a & 0 & 0 & b \\ 0 & 0 & a & 0 & b \\ 0 & 0 & 0 & a & b \\ b & b & b & b & c \end{pmatrix}$$

$$[X] = \begin{bmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \\ \ddot{\phi}_3 \\ \ddot{\phi}_4 \\ \ddot{\theta}' \end{bmatrix} \qquad [B] = \begin{bmatrix} -p\ \phi_1 \\ -p\ \phi_2 \\ -p\ \phi_3 \\ -p\ \phi_4 \\ 0 \end{bmatrix}$$

where
$$\dot{\theta} = \omega_o + \dot{\theta}', \quad \ddot{\theta} = \ddot{\theta}'$$
$$a = mr^2 + \rho r^3/3$$
$$b = mr^2 + \rho r^3/3 + \rho r_o (mr + \rho r^2/2)$$
$$c = I_o + \sum_{i=1}^{4} [mr^2 + \rho r^{3/3} + 2r_o (mr + \rho r^{2/2} + \rho r_o^2 (m + \rho r)]$$
$$= I_T$$
$$p = (mr + \rho r^2/2)\ r_o\ \omega_o^2$$

Since we expect the motion to be oscillatory, we attempt solutions of the form:-

$$\phi_i(t) = \phi_i (t=0)\ e^{i\omega t}$$

and

$$\theta'(t) = \theta'(t=0)\ e^{i\omega t}$$

where the frequency ω is to be determined.
Substituting the solutions ϕ_i (t) and θ' (t) into the matrix equation of motion AX = B, we find

$$\begin{bmatrix} a & 0 & 0 & 0 & b \\ 0 & a & 0 & 0 & b \\ 0 & 0 & a & 0 & b \\ 0 & 0 & 0 & a & b \\ b & b & b & b & c \end{bmatrix} \begin{bmatrix} \omega^2\ \phi_1 \\ \omega^2\ \phi_2 \\ \omega^2\ \phi_3 \\ \omega^2\ \phi_4 \\ \omega^2\ \theta' \end{bmatrix} = \begin{bmatrix} p\ \phi_1 \\ p\ \phi_2 \\ p\ \phi_3 \\ p\ \phi_4 \\ 0 \end{bmatrix}$$

This equation is just a set of five simultaneous equations:

$$\begin{pmatrix} a\omega^2-p & 0 & 0 & 0 & b\omega^2 \\ 0 & a\omega^2-p & 0 & 0 & b\omega^2 \\ 0 & 0 & a\omega^2-p & 0 & b\omega^2 \\ 0 & 0 & 0 & a\omega^2-p & b\omega^2 \\ b\omega^2 & b\omega^2 & b\omega^2 & b\omega^2 & c\omega^2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \theta' \end{pmatrix} = 0 \tag{4-2}$$

In order that nontrivial solutions exist, the characteristic determinant must vanish. Therefore,

$$\det \begin{vmatrix} a\omega^2-p & 0 & 0 & 0 & b\omega^2 \\ 0 & a\omega^2-p & 0 & 0 & b\omega^2 \\ 0 & 0 & a\omega^2-p & 0 & b\omega^2 \\ 0 & 0 & 0 & a\omega^2-p & b\omega^2 \\ b\omega^2 & b\omega^2 & b\omega^2 & b\omega^2 & c\omega^2 \end{vmatrix} = 0$$

The determinant can be calculated by using Laplace's expansion. The result is:-

$$\omega^2 (a\omega^2 - p)^3 \; [(a\omega^2 - p)\, c - 4 b^2 \omega^2] = 0$$

This polynomial equation yields three distinct roots: ω_1, ω_2, ω_3, viz.,

$$\omega_1 = 0, \quad \omega_2 = \sqrt{\frac{p}{a}}, \quad \omega_3 = \sqrt{\frac{p}{a}} \sqrt{\frac{1}{1-4b^2/(ac)}}$$

where ω_2 is a triple root.

These are the frequencies of three distinct modes of oscillations described by the five variables (ϕ_1, ϕ_2, ϕ_3, ϕ_4, θ'). The triple root corresponds to three possible patterns of boom motion in such a manner that the total boom angular momentum is zero. The third distinct root corresponds to coupled hub-booms oscillations.

$$\lim_{\rho\to 0} \omega_2 = \omega_o \sqrt{\frac{r_o}{r}} \quad \text{(uncoupled)} \qquad \text{[Triple]}$$

$$\lim_{\rho\to 0} \omega_3 = \omega_o \sqrt{\frac{r_o}{r} \frac{I_T}{I_o}} \quad \text{(coupled)}$$

The hub-booms patterns of motion (modes) can be found by substituting the frequencies into the set of simultaneous equations of motion (equ.4-1). We find

For $\omega = \omega_1$,	$\phi = 0$,	$\theta' =$ arbitrary	(Mode 1)	
For $\omega = \omega_2$,	$\sum_{i=1}^{4} \phi_i = 0$	$\theta' = 0$	(Mode 2)	[Triple]
For $\omega = \omega_3$,	$\phi_i = \frac{I_T \theta'}{4b}$	$i=i, ---4$	(Mode 3)	

4.2 Harmonic Approximation with Translation

In harmonic normal mode analysis, all the responses to excitations of modes are infinitesimal. This is why we have separated θ (t) (hub angular velocity) into two parts: ω_0 and $\theta'(t)$, the latter being the infinitesimal response to mode excitation, where the former ($\theta'_0 = \omega_0$) can be arbitrarily large and is not a part of mode excitation itself.

Likewise, the translations X and Y should be written as $X = X_0 + X'$ and $Y = Y + Y'$, the dashed variables being the infinitesimal response to excitation of modes. The condition or constraint that the center of mass remains stationary is

$$MX + \sum_{i=1}^{4} (m + \rho r_i) \ (X + \cos \ \Phi_i) = 0$$

$$\text{where } \Phi_i = \theta(t) + \phi_i(t) + (i-1)\frac{\pi}{2}$$

Let $X = X_0 + X'$, where

$$\left[M + \sum_{i=1}^{4} (m + \rho r_i) \right] X_o = -\sum_{i=1}^{4} \left(mr_i + \rho \frac{r_i^{\,2}}{2} \right) \cos \Phi_i \Bigg|_{\phi_i = \text{const}}$$

and $X' = X - X_o$

The time variable in X_o is $\theta(t)$ where ϕ_i = constant, so that X_o is due to hub rotation and not mode excitation, while X' is the infinitesimal response of the order $O(\epsilon)$, as ϕ, θ' are. Thus, the inplane oscillation variables are ϕ_1, ϕ_2, ϕ_3, ϕ_4, θ', X, and Y; There are 7 inplane variables, and there must be 7 natural modes.

From the equations of motion including translation, we have the following matrix equation of normal mode oscillation in harmonic approximation:

$$\begin{pmatrix} a & 0 & 0 & 0 & b & -d & e \\ 0 & a & 0 & 0 & b & -e & -d \\ 0 & 0 & a & 0 & b & d & -e \\ 0 & 0 & 0 & a & b & e & d \\ b & b & b & b & c & 0 & 0 \\ -d & -e & d & e & 0 & m & 0 \\ e & -d & -e & d & 0 & 0 & m \end{pmatrix} \begin{pmatrix} -\omega^2 & \phi_1 \\ -\omega^2 & \phi_2 \\ -\omega^2 & \phi_3 \\ -\omega^2 & \phi_4 \\ -\omega^2 & \theta' \\ -\omega^2 & X' \\ -\omega^2 & Y' \end{pmatrix} = \begin{pmatrix} -p\phi_1 \\ -p\phi_2 \\ -p\phi_2 \\ -p\phi_4 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad (4\text{-}3)$$

where terms of the order $O(\epsilon^2)$, $O(\epsilon/\mathcal{M})$ or higher are neglected. The notations are as follows:

$$a = mr^2 + \rho r^3/3$$

$$b = mr^2 + \rho r^3/3 + r_0 (mr + \rho r^2/2)$$

$$c = I_T = I_o + \sum_{i=1}^{4} \left[mr^2 + \rho \frac{r^3}{3} + 2 r_o (mr + \rho \frac{r^3}{3}) + r_o^2 (m + \rho r) \right]$$

$$d = (mr + \rho \frac{r^2}{2}) \sin\theta$$

$$e = (mr + \frac{r^2 \rho}{2}) \cos\theta$$

$$\mathcal{M} = M + \sum_{i=1}^{4} m_i = M + 4m$$

$$p = (mr + \rho r^2/2)\, r_o\, \omega_o^2$$

In order that nontrivial solutions exist, the characteristic determinant of the above matrix equation must vanish: det $| A | = 0$

Therefore,

$$\det \begin{vmatrix} a\omega^2-p & 0 & 0 & 0 & b\omega^2 & -d\omega^2 & e\omega^2 \\ 0 & a\omega^2-p & 0 & 0 & b\omega^2 & -e\omega^2 & -d\omega^2 \\ 0 & 0 & a\omega^2-p & 0 & b\omega^2 & d\omega^2 & -e\omega^2 \\ 0 & 0 & 0 & a\omega^2-p & b\omega^2 & e\omega^2 & d\omega^2 \\ b\omega^2 & b\omega^2 & b\omega^2 & b\omega^2 & c\omega^2 & 0 & 0 \\ -d\omega^2 & -e\omega^2 & d\omega^2 & e\omega^2 & 0 & \mathcal{M}\omega^2 & 0 \\ e\omega^2 & -d\omega^2 & -e\omega^2 & d\omega^2 & 0 & 0 & \mathcal{M}\omega^2 \end{vmatrix} = 0$$

Using Laplace expansion (Appendix D), we find the determinant gives:

$$\omega^6 \left[\mathcal{M}(a\omega^2-p) \; -2\omega^2 \left(mr + \rho\frac{r^2}{2}\right)^2\right]^2 (a\omega^2-p) \left[(a\omega^2-p)\, c \; -4b^2\omega^2\right] = 0$$

The roots of this polynomial equation are -

$$\omega_1 = 0$$

$$\omega_2 = 0$$

$$\omega_3 = 0$$

$$\omega_4 = \sqrt{\frac{p}{a}}$$

$$\omega_5 = \sqrt{\frac{p}{a}} \sqrt{\frac{1}{1-\dfrac{2(mr+\rho r^2/2)^2}{\mathcal{M}a}}}$$

$$\omega_6 = \omega_5$$

$$\omega_7 = \sqrt{\frac{p}{a}} \sqrt{\frac{1}{1-4b^2/(ac)}}$$

Substituting the frequencies ω_i (eigenvalues) into the matrix equation of motion, one obtains the corresponding eigenfunctions, which can be put into a matrix form [B].

$$B = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 & -1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & -\frac{4b}{c} \\ 0 & 1 & 0 & 0 & F & -G & 0 \\ 0 & 0 & 1 & 0 & -G & -F & 0 \end{pmatrix} \tag{4-4}$$

where $F = \dfrac{2(d+e)}{m}$

$G = \dfrac{2(e-d)}{m}$

One readily verifies the following matrix relation:

$$B^T A B = \operatorname{diag}(\omega_1, \ldots, \omega_7)$$

Explicitly,

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & F & -G \\ 1 & -1 & -1 & 1 & 0 & -G & -F \\ 1 & 0 & 1 & 1 & \frac{4b}{c} & 0 & 0 \end{pmatrix} \begin{pmatrix} a\omega^2-p & 0 & 0 & 0 & b & -d & e \\ 0 & a\omega^2-p & 0 & 0 & b & -e & -d \\ 0 & 0 & a\omega^2-p & 0 & b & d & -e \\ 0 & 0 & 0 & a\omega^2-p & b & e & d \\ b & b & b & b & c & 0 & 0 \\ -d & -e & d & e & 0 & n & 0 \\ e & -d & -e & d & 0 & 0 & n \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & -\frac{4b}{c} \\ 0 & 1 & 0 & 0 & -G & 0 \\ 0 & 0 & 1 & 0 & -F & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \omega_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega_7 \end{pmatrix} \tag{4-5}$$

where ω_i have been given on the previous page. There are three zero frequencies, three non-zero frequencies, and two doubly-degenerate frequencies. The natural modes and their frequencies for in-plane dynamics are displayed in Figure 3.

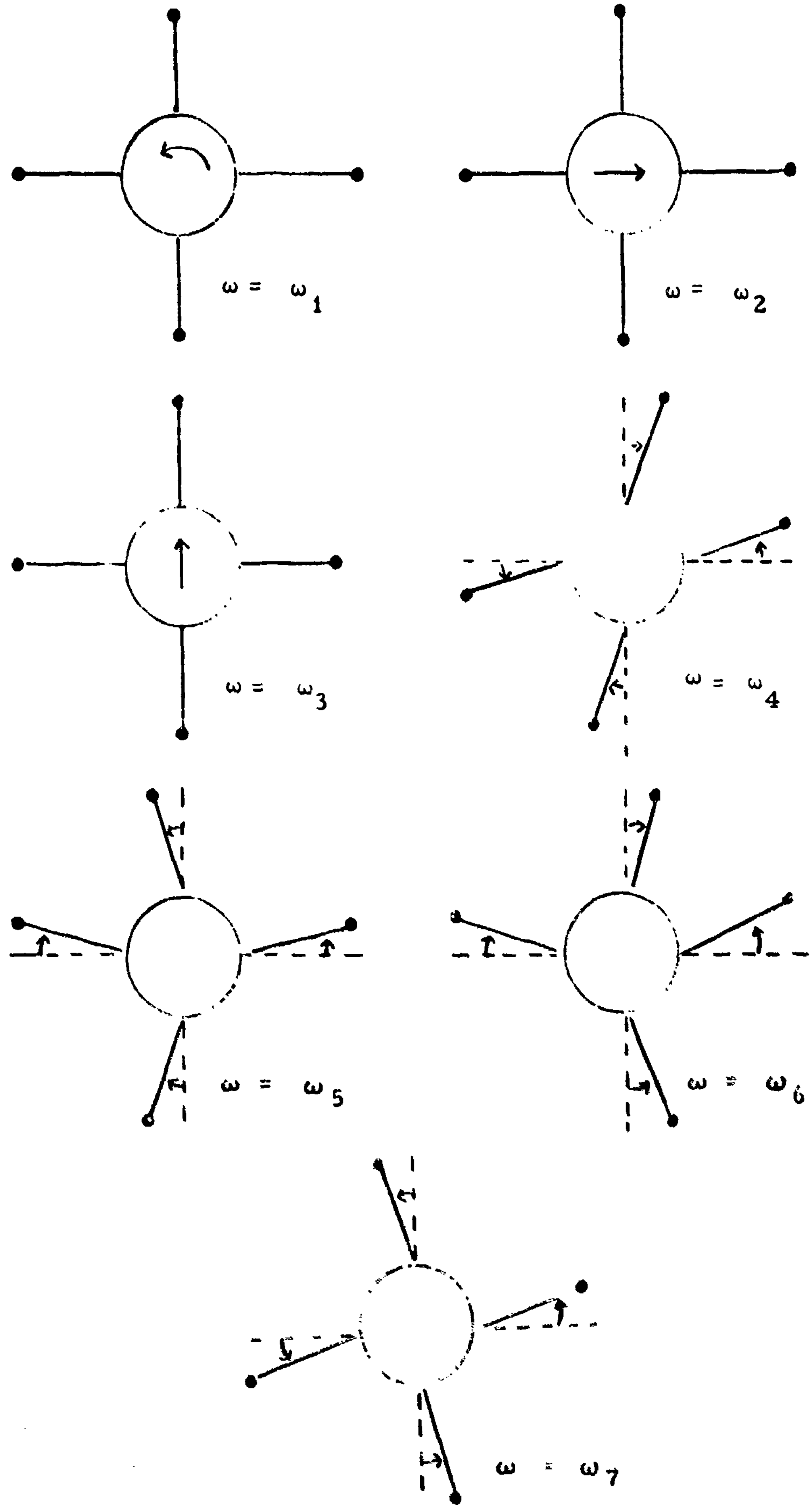


Figure 3. Inplane Normal Modes for Equal Boom Lengths

CHAPTER 5

NORMAL COORDINATES FOR INPLANE OSCILLATIONS

5.1 Harmonic Approximation with Matrix Formulation

In harmonic approximation, with no deployment/retraction, no damping and no spring constant, the full Lagrangian L as given on page 31(Chapter 3) becomes simplified as follows:

$$L = L_o + \sum_{i=1}^{4} (L_i + L_i') \qquad (5\text{-}1)$$

where

$$L_o = \frac{1}{2} I_o \dot{\theta}^2 + \frac{1}{2} M (\dot{X}^2 + \dot{Y}^2)$$

$$L_i = \frac{1}{2}(mr_i^2 + \rho\frac{r_i^3}{3}) \dot{\phi}_i^2 + \dot{\theta}[(mr_i^2 + \rho\frac{r_i^3}{3}) \dot{\phi}_i + (mr_i + \rho\frac{r_i^2}{2}) r_o \cos\phi_i \dot{\phi}_i] + \frac{1}{2}\dot{\theta}^2 [(mr_i^2 + \rho\frac{r_i^3}{3}) + (m + \rho r_i) r_o^2 + 2(mr_i + \rho\frac{r_i^2}{2}) r_o \cos \phi_i]$$

$$L_i' = \frac{1}{2}(m + \rho r_i)(\dot{X}^2 + \dot{Y}^2) + \dot{\theta}[(m+\rho r_i)(r_o \cos\Theta_i \dot{Y} - r_o \sin\Theta_i \dot{X}) + (mr_i + \rho\frac{r_i^2}{2}) (\dot{Y}\cos\phi_i - \dot{X}\sin\phi_i)] + (mr_i + \rho\frac{r_i^2}{2}) \dot{\phi}_i (-\dot{X}\sin\phi_i + \dot{Y}\cos\phi_i)$$

Further simplicication of the harmonic approximated Lagrangian is achieved by observing that:

$$\sum_{i=1}^{4} \cos\Theta_i = \sum_{i=1}^{4} \cos (0 + (i-1)\frac{\pi}{2}) = 0$$

$$\sum_{i=1}^{4} \sin\Theta_i = \sum_{i=1}^{4} \sin (0 + (i-1)\frac{\pi}{2}) = 0$$

and

$$I_T = \sum_{i=1}^{4} [(mr_i^2 + \rho\frac{r_i^3}{3}) + (mr + \rho r_i)r_o^2 + 2(mr_i + \rho\frac{r_i^2}{2}) r_o] + I_o$$

which is a constant in case of no deployment/retraction. Using $\cos\phi_i \doteq 1-\phi_i^2/2$ in harmonic approximated, one finds a simplified Lagrangian L as given below:

$$L = \frac{1}{2} I_T \dot\theta^2 + \frac{1}{2}(M + \sum_{i=1}^{4}(m+\rho r_i))(\dot X^2 + \dot Y^2) + \sum_{i=1}^{4}\Big\{ \frac{1}{2}(m r_i^2 + \rho\frac{r_i^3}{3})\ \dot\phi_i^2 + [(m r_i^2 + \rho\frac{r_i^3}{3}) + (m r_i + \rho\frac{r_i^2}{2}) r_o]\dot\theta\ \dot\phi_i - \frac{1}{2}\dot\theta^2 (m r_i + \rho\frac{r_i^2}{2}) r_o \frac{\phi_i^2}{2}\Big\} + \sum_{i=1}^{4}\Big\{ (m r_i + \rho\frac{r_i^2}{2})(\cos\Phi_i\,\dot\theta\,\dot Y - \sin\Phi_i\,\dot\theta\,\dot X) + (m r_i + \rho\frac{r_i^2}{2})(\cos\Phi_i\,\dot\phi_i\,\dot Y - \sin\Phi_i\,\dot\phi_i\,\dot X)\Big\} \quad (5\text{-}2)$$

Neglecting from the Lagrangian all constant terms, which do not contribute to normal modes, we find the Langrangian is :

$$L = \frac{1}{2} I_T \dot\theta'^2 + \frac{1}{2}\mathcal{M}(\dot X^2 + \dot Y^2) + \frac{1}{2}\sum_{i=1}^{4}\Big\{(m r_i^2 + \rho\frac{r_i^3}{3})\ \dot\phi_i^2 + 2[m r_i^2 + \rho\frac{r_i^3}{3} + (m r_i + \rho\frac{r_i^2}{2}) r_o]\dot\theta'\dot\phi_i - \omega_o^2 (m r_i + \rho\frac{r_i^2}{2}) r_o\ \phi_i^2\Big\} + \frac{1}{2}\sum_{i=1}^{4}\Big\{2(m r_i + \rho\frac{r_i^2}{2})(\cos\Phi_i\,\dot\phi_i\,\dot Y - \sin\Phi_i\,\dot\phi_i\,\dot X)\Big\} + \sum_{i=1}^{4}\Big\{(m r_i + \rho\frac{r_i^2}{2})(\cos\Phi_i\,\dot\theta\,\dot Y - \sin\Phi_i\,\dot\theta\,\dot X)\Big\} \quad (5\text{-}3)$$

It should be reminded that $\dot\theta = \omega_o + \dot\theta'$ and that $\dot X$ and $\dot Y$ need not be as small as $\dot\phi_i$ because $\dot X$ and $\dot Y$ are functions of ω_o and $\dot\phi_i$. From conservation of linear momenta (Appendix A), we have:

$$\dot X = \frac{\sum_{i=1}^{4}(m r_i + \rho\frac{r_i^2}{2})(\omega_o + \dot\theta' + \dot\phi_i)\sin\Phi_i}{M + \sum_{i=1}^{4}(m + \rho r_i)}$$

$$\dot Y = \frac{-\sum_{i=1}^{4}(m r_i + \rho\frac{r_i^2}{2})(\omega_o + \dot\theta' + \dot\phi_i)\cos\Phi_i}{M + \sum_{i=1}^{4}(m + \rho r_i)}$$

In harmonic normal mode analysis, all the responses to excitations of modes are infinitesimal. This is why we have separated $\dot{\theta}$ (hub angular velocity) into two parts, ω_o and θ', the latter being the infinitesimal response to mode excitation, while the former ($\dot{\theta}_o = \omega_o$) can be arbitrarily large and is not a part of mode excitation itself. Likewise, the translations X and Y should be written as $\dot{X} = \dot{X}_o + \dot{X}'$ and $\dot{Y} = \dot{Y}_o + \dot{Y}'$, the dashed variable being the infinitesimal excitation. By considering that the center of mass remains stationary, we have:

$$[M + \sum_{i=1}^{4} (m + \rho r_i)] X = - \sum_{i=1}^{4} (m r_i + \rho r_i^2/2) \cos \Phi_i$$

where $\quad \Phi_i = \theta(t) + \phi_i(t) + (i-1)\frac{\pi}{2}$

Let $\quad X = X_o + X'$, where

$$[M + \sum_{i=1}^{4} (m + \rho r_i)] X_o = - \sum_{i=1}^{4} (m r_i + \rho r_i^2/2) \cos \Phi_i \Big|_{\phi_i = \text{const}}$$

$$X' = X - X_o \qquad (5\text{-}4)$$

The time varying quantity in X_o is $\theta(t)$ only, so that X_o is not due to mode excitation, while X' is the infinitesimal response, as ϕ_i, θ' are infintesimal $\sim 0\ (\epsilon)$. Therefore, in harmonic normal mode analysis, the mode variables are ϕ_1, ϕ_2, ϕ_3, ϕ_4, θ', X', and Y' as far as in-plane modes are concerned.

Subtracting the part due to constant hub rotation ω_o, we have:

$$\dot{X}' = \frac{\sum_{i=1}^{4} (m r_i + \rho \frac{r_i^2}{2}) (\dot{\theta}' + \dot{\phi}_i) \sin \Phi_i}{M + \sum_{i=1}^{4} (m + \rho r_i)}$$

$$\dot{Y}' = \frac{- \sum_{i=1}^{4} (m r_i + \rho \frac{r_i^2}{2}) (\dot{\theta}' + \dot{\phi}_i) \cos \Phi_i}{M + \sum_{i=1}^{4} (m + \rho r_i)}$$

As a result, the last term of the harmonic approximated Lagrangian as given on page 50 does not contribute to the kinetic energy, for,

$$L_{(\text{last term})} \sim \sum_{i=1}^{4} \left\{ (m r_i + \rho \frac{r_i^2}{2}) (\dot{\theta}\, \dot{Y} \cos \phi_i - \dot{\theta} \dot{X} \sin \phi_i) \right\}$$

$$\sim \sum_{i=1}^{4} \left\{ (m r_i + \rho \frac{r_i^2}{2}) (\dot{\theta}' \dot{Y}' \cos \phi_i + \dot{\theta}' \dot{X}' \sin \phi_i) \right\}$$

$$+ \sum_{i=1}^{4} \left\{ (m r_i + \rho \frac{r_i^2}{2}) (\omega_o \dot{Y}_o \cos \phi_i - \omega_o \dot{X}_o \sin \phi_i) \right\}$$

+ terms not contributing to normal modes

Since $\Sigma \cos \phi_i \sim O(\epsilon)$ and $\Sigma \sin \phi_i \sim O(\epsilon)$, the first sum on the R.H.S. is of the order $O(\epsilon^3)$, and is therefore neglegible. The second sum, however is of the order $O(\epsilon^2/\mathcal{M})$, and, for equal boom lengths case, becomes

$$L_{(\text{last term})} \sim -\frac{\omega_o^2}{\mathcal{M}} \sum_{i,j=1}^{4} \left\{ (m r_i + \rho \frac{r_i^2}{2})^2 \left[\cos \phi_i \cos \phi_j - \sin \phi_i \sin \phi_j \right] \right\}$$

$$\sim -\frac{\omega_o^2}{2\mathcal{M}} \left\{ \sum_{i=1}^{2} (m r_i + \rho \frac{r_i^2}{2})^2 (\phi_i - \phi_{i+2})^2 \right\}$$

Thus, the harmonic approximated Lagrangian becomes:

$$L = \frac{1}{2} I_T \dot{\theta}'^2 + \frac{1}{2} \mathcal{M} (\dot{X}'^2 + \dot{Y}'^2) + \frac{1}{2} \sum_{i=1}^{4} \Big\{ (m r_i^2 + \rho \frac{r_i^3}{3}) \dot{\phi}_i^2 +$$

$$2 \left[(m r_i^2 + \rho \frac{r_i^3}{3}) + r_o (m r_i + \rho \frac{r_i^2}{2}) \right] \dot{\theta}' \dot{\phi}_i + 2 (m r_i + \rho \frac{r_i^2}{2})$$

$$(\cos \phi_i \dot{\phi}_i \dot{Y}' - \sin \phi_i \dot{\phi}_i \dot{X}') \Big\} + \frac{1}{2} \sum_{i=1}^{4} \left\{ -\omega_o^2 (m r_i + \rho \frac{r_i^2}{2}) r_o \dot{\phi}_i^2 \right\}$$

$$- \frac{1}{2} \frac{\omega_o^2}{\mathcal{M}} \sum_{i=1}^{2} \left\{ (m r_i + \rho \frac{r_i^2}{2})^2 (\phi_i - \phi_{i+2})^2 \right\}$$

In matrix notation, the Lagrangian in harmonic approximation is:

$$L = \frac{1}{2} \left(\dot\phi_1 \;\; \dot\phi_2 \;\; \dot\phi_3 \;\; \dot\phi_4 \;\; \dot\theta' \;\; \dot X' \;\; \dot Y' \right) \begin{pmatrix} a & 0 & 0 & 0 & b & -d & e \\ 0 & a & 0 & 0 & b & -e & -d \\ 0 & 0 & a & 0 & b & d & -e \\ 0 & 0 & 0 & a & b & e & d \\ b & b & b & b & c & 0 & 0 \\ -d & -e & d & e & 0 & \mathcal{M} & 0 \\ e & -d & -e & d & 0 & 0 & \mathcal{M} \end{pmatrix} \begin{pmatrix} \dot\phi_1 \\ \dot\phi_2 \\ \dot\phi_3 \\ \dot\phi_4 \\ \dot\theta' \\ \dot X' \\ \dot Y' \end{pmatrix}$$

$$- \frac{1}{2} \left(\phi_1 \;\; \phi_2 \;\; \phi_3 \;\; \phi_4 \;\; \theta' \;\; X' \;\; Y' \right) \begin{pmatrix} \bar p & 0 & -q & 0 & 0 & 0 & 0 \\ 0 & \bar p & 0 & 0 & 0 & 0 & 0 \\ -q & 0 & \bar p & -q & 0 & 0 & 0 \\ 0 & -q & 0 & \bar p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \theta' \\ X' \\ Y' \end{pmatrix}$$

where

$$a = (mr^2 + \rho \frac{r^3}{3})$$

$$b = (mr^2 + \rho \frac{r^3}{3}) + (mr + \rho \frac{r^2}{2})\, r_o$$

$$c = I_T$$

$$d = (mr + \rho \frac{r^2}{2}) \sin \omega_o t$$

$$e = (mr + \rho \frac{r^2}{2}) \cos \omega_o t$$

$$\mathcal{M} = M + 4\,(m + \rho r)$$

$$p = (mr + \rho \frac{r^2}{2})\, r_o \;\; \omega_o^2$$

$$q = (mr + \rho r^2/2)^2\, \omega_o^2 / \mathcal{M}$$

$$\bar p = p + q$$

The Langrangian L has been written in such a form that the matrices [T] and [V] are manifested.

$$L = \frac{1}{2}\,[\,\dot{\phi}_1 \cdots \dot{Y}'\,][T]\begin{bmatrix}\dot{\phi}_1\\ \vdots\\ \dot{Y}'\end{bmatrix} - \frac{1}{2}\,[\,\phi_1 \cdots Y'\,]\;[V]\;\begin{bmatrix}\phi_1\\ \vdots\\ Y'\end{bmatrix}$$

The orthogonal matrix is

$$[B] = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 & -1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & -\frac{4b}{c} \\ 0 & 1 & 0 & 0 & F & -G & 0 \\ 0 & 0 & 1 & 0 & -G & -F & 0 \end{bmatrix} \qquad (5\text{-}5)$$

where $F = \frac{2(d+e)}{m}$

$G = \frac{2(-d+e)}{m}$

5.2 Orthogonal Transformation of Kinetic Energy

[B] is not orthogonal in the usual sense:

ie. $[B]^T\,[B] \neq I$

[B] should be orthogonal in the unusual sense:

$$[B]^T\,[T]\,[B] = I$$

$$[T]\,[B] = \begin{bmatrix} a & 0 & 0 & 0 & b & -d & e \\ 0 & a & 0 & 0 & b & -e & -d \\ 0 & 0 & a & 0 & b & d & -e \\ 0 & 0 & 0 & a & b & e & d \\ b & b & b & b & c & 0 & 0 \\ -d & -e & d & e & 0 & \mathcal{M} & 0 \\ e & -d & -e & d & 0 & 0 & \mathcal{M} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 & -1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & -\frac{4b}{c} \\ 0 & 1 & 0 & 0 & F & -G & 0 \\ 0 & 0 & 1 & 0 & -G & -F & 0 \end{bmatrix}$$

$$= \begin{bmatrix} b & -d & e & a & a-dF-eG & a+dG-ef & a-\frac{4b^2}{c} \\ b & -e & -d & -a & a-eF+dG & -a+eG+dF & a-\frac{4b^2}{c} \\ b & d & -e & a & -a+dF+eG & -a-dG+eF & a-\frac{4b^2}{c} \\ b & e & d & -a & -a+eF-dG & a-eG-dF & a-\frac{4b^2}{c} \\ c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{M} & 0 & 0 & -2(d+e)+\mathcal{M}F & -2(d-e)-G\mathcal{M} & 0 \\ 0 & 0 & \mathcal{M} & 0 & -2(d-e)-G\mathcal{M} & 2(d+e)-\mathcal{M}F & 0 \end{bmatrix}$$

$$[B]^T\,[T]\,[B] =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & F & -G \\ 1 & -1 & -1 & 1 & 0 & -G & -F \\ 1 & 1 & 1 & 1 & -\frac{4b}{c} & 0 & 0 \end{bmatrix} \begin{bmatrix} b & -d & e & a & a-dF\ eG & a+dG-eF & a-\frac{4b^2}{c} \\ b & -e & -d & -a & a-eF+dG & -a+eG+dF & a-\frac{4b^2}{c} \\ b & d & -e & +a & -a+dF+eG & -a-dG+dF & a-\frac{4b^2}{c} \\ b & e & d & -a & -a+eF-dG & a-eG-dF & a-\frac{4b^2}{c} \\ c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{M} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{M} & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{M} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{M} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & D & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & D & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4(a-\frac{4b^2}{c}) \end{bmatrix}$$

where $D = 4a - 2(d+e)F + 2(d-e)G = 4a\left[1 - \dfrac{2(d^2+e^2)}{a\,\mathcal{M}}\right]$

$$= 4\left[a - \frac{2(mr + \rho\frac{r^2}{2})^2}{\mathcal{M}}\right]$$

The above result of $[B]^T[T][B]$ is not unity matrix because B has not been normalized. It would be unity if B were normalized; viz., normalized [B] =

$$\begin{bmatrix} 0 & 0 & 0 & \beta & \nu & \nu & \gamma \\ 0 & 0 & 0 & -\beta & \nu & -\nu & \gamma \\ 0 & 0 & 0 & \beta & -\nu & -\nu & \gamma \\ 0 & 0 & 0 & -\beta & -\nu & \nu & \gamma \\ \alpha & 0 & 0 & 0 & 0 & 0 & -\gamma\frac{4b}{c} \\ 0 & \mu & 0 & 0 & F\nu & -G\nu & 0 \\ 0 & 0 & \mu & 0 & -G\nu & -F\nu & 0 \end{bmatrix} \qquad (5\text{-}6)$$

where

$$\alpha = \frac{1}{\sqrt{c}}$$

$$\beta = \frac{1}{2\sqrt{a}}$$

$$\gamma = \frac{1}{2\sqrt{a - \frac{4b^2}{c}}}$$

$$\mu = \frac{1}{\sqrt{\mathcal{M}}}$$

$$\nu = \frac{1}{2\sqrt{a - \dfrac{2(mr + \rho\frac{r^2}{2})^2}{M + 4m}}}$$

With not much confusion hopefully, let us still use [B] to denote normalized [B]. Now, this normalized [B] satisfies the unusual orthogonality relation:

$$[B]^T\,[T]\,[B] = [I] \qquad (5\text{-}7)$$

Now, we can define a new set of coordinates $\{\xi_i\}$ related to the original coodinates $\{\phi_i, \theta, X', Y'\}$ by the equation:

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \theta' \\ X' \\ Y' \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 & \beta & \nu & \nu & \gamma \\ 0 & 0 & 0 & -\beta & \nu & -\nu & \gamma \\ 0 & 0 & 0 & \beta & -\nu & -\nu & \gamma \\ 0 & 0 & 0 & -\beta & -\nu & \nu & \gamma \\ \alpha & 0 & 0 & 0 & 0 & 0 & \gamma(-\frac{4h}{c}) \\ 0 & \mu & 0 & 0 & F\nu & -G\nu & 0 \\ 0 & 0 & \mu & 0 & -G\nu & -F\nu & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \\ \xi_7 \end{bmatrix} \qquad (5\text{-}8)$$

where $\{\xi_i\}$ are the normal coordinates.

Limit : $\rho \to 0$

$$a \to mr^2$$

$$b \to mr(r + r_o)$$

$$c \to I_T$$

$$d \to mr \sin \omega_o t$$

$$e \to mr \cos \omega_o t$$

$$\mathcal{M} \to M + 4m$$

$$\alpha \to \frac{1}{\sqrt{I_T}} = \frac{1}{\sqrt{I_o + 4m(r + r_o)^2}}$$

$$\beta \to \frac{1}{2\sqrt{mr}}$$

$$\gamma \to \frac{1}{2\sqrt{mr}} \sqrt{\frac{I_T}{I_o}} = \beta \sqrt{\frac{I_T}{I_o}}$$

$$\mu \to \frac{1}{\sqrt{M + 4m}}$$

$$\nu \to \frac{1}{2\sqrt{mr^2 - \frac{2m^2 r^2}{(M + 4m)}}} = \frac{1}{2r} \frac{1}{\sqrt{\frac{Mm + 2m^2}{M + 4m}}} = \frac{1}{2r}\sqrt{\frac{M + 4m}{Mm + 2m^2}}$$

$$= \frac{1}{2\sqrt{mr}} \sqrt{\frac{M + 4m}{M + 2m}} \simeq \beta\left(1 + \frac{m}{M} + 0\left(\frac{m^2}{M^2}\right)\right)$$

for small $\frac{m}{M} \ll 1$

The kinetic energy K. E. in terms of these normal coordinates is:

$$K.E. = \frac{1}{2}[\dot{\xi}]^T [B]^T [T] [B] [\dot{\xi}]$$

But the unusual orthogonality relation is:

$$[B]^T [T] [B] = [I]$$

Hence, the kinetic energy in normal coordinates becomes:

$$\boxed{K.E. = \frac{1}{2} \sum_{i=1}^{7} \dot{\xi}_i \dot{\xi}_i} \qquad (5\text{-}9)$$

where $$[\xi] = [B]^{-1} [\phi]$$

The inverse $[B]^{-1}$ of $[B]$ is calculated in Appendix F. It is

$$[B]^{-1} = \begin{bmatrix} \frac{b}{c\alpha} & \frac{b}{c\alpha} & \frac{b}{c\alpha} & \frac{b}{c\alpha} & \frac{1}{\alpha} & 0 & 0 \\ -\frac{F-G}{4\mu} & -\frac{F+G}{4\mu} & \frac{F-G}{4\mu} & \frac{F+G}{4\mu} & 0 & \frac{1}{\mu} & 0 \\ \frac{F+G}{4\mu} & -\frac{F-G}{4\mu} & -\frac{F+G}{4\mu} & \frac{F-G}{4\mu} & 0 & 0 & \frac{1}{\mu} \\ \frac{1}{4\beta} & -\frac{1}{4\beta} & \frac{1}{4\beta} & -\frac{1}{4\beta} & 0 & 0 & 0 \\ \frac{1}{4\nu} & \frac{1}{4\nu} & -\frac{1}{4\nu} & -\frac{1}{4\nu} & 0 & 0 & 0 \\ \frac{1}{4\nu} & -\frac{1}{4\nu} & -\frac{1}{4\nu} & \frac{1}{4\nu} & 0 & 0 & 0 \\ \frac{1}{4\gamma} & \frac{1}{4\gamma} & \frac{1}{4\gamma} & \frac{1}{4\gamma} & 0 & 0 & 0 \end{bmatrix}$$

where $$\frac{b}{c\alpha} = \frac{(mr^2 + \rho\frac{r^3}{3}) + r_o (mr + \rho\frac{r^2}{2})}{\sqrt{I_T}}$$

$$\frac{F+G}{4\mu} = \frac{e}{\mathcal{M}\mu} = \frac{(mr + \rho r^2/2) \cos \omega_o t}{\sqrt{M + 4(m + \rho r)}}$$

$$\frac{F-G}{4\mu} = \frac{d}{\mathcal{M}\mu} = \frac{(mr + \rho r^2/2) \sin \omega_o t}{\sqrt{M + 4(m + \rho r)}}$$

$$\frac{1}{4\beta} = \frac{\sqrt{a}}{2} = \frac{1}{2}\sqrt{mr^2 + \rho\frac{r^3}{3}}$$

$$\frac{1}{4\nu} = \frac{1}{2}\sqrt{\left(mr^2 + \rho\frac{r^3}{3}\right) - \frac{2\left(mr + \rho\frac{r^2}{2}\right)^2}{M + 4(m + \rho r)}}$$

$$\frac{1}{4\gamma} = \frac{1}{2}\sqrt{\left(mr^2 + \rho\frac{r^3}{3}\right) - \frac{4\left[\left(mr^2 + \rho r^3/3\right) + \left(mr + \quad r^2/2\right) r_o\right]^2}{I_T}}$$

Limiting Values:

Limit $\rho \longrightarrow 0$ $\quad \frac{b}{c\,\alpha} \longrightarrow \frac{mr\,(r + r_o)}{\sqrt{I_T}}$

$$\frac{F + G}{4\mu} \longrightarrow \frac{mr\cos\,\omega_o t}{\sqrt{M + 4m}}$$

$$\frac{F - G}{4\mu} \longrightarrow \frac{mr\,\sin\omega_o t}{\sqrt{M + 4m}}$$

$$\frac{1}{4\beta} \longrightarrow \frac{\sqrt{m}\;r}{2}$$

$$\frac{1}{4\nu} \longrightarrow \frac{1}{2}\sqrt{mr^2 - \frac{2m^2r^2}{M + 4m}} = \frac{1}{2}\sqrt{\frac{Mmr^2 + 2m^2r^2}{M + 4m}}$$

$$= \frac{\sqrt{m}\;r}{2}\sqrt{\frac{M + 2m}{M + 4m}}$$

$$\frac{1}{4\gamma} \longrightarrow \frac{1}{2}\sqrt{mr^2 - \frac{4m^2r^2\,(r + r_o)^2}{I_T}} = \frac{\sqrt{m}\,r}{2}\sqrt{\frac{I_o}{I_T}}$$

Therefore, the normal coordinates are:

$$
\begin{aligned}
\xi_1 &= \frac{b}{c\alpha}(\phi_1 + \phi_2 + \phi_3 + \phi_4) + \frac{1}{\alpha}\theta' \\
\xi_2 &= \frac{1}{4\mu}[-(F-G)\phi_1 - (F+G)\phi_2 + (F-G)\phi_3 + (F+G)\phi_4] + \frac{1}{\mu}X \\
\xi_3 &= \frac{1}{4\mu}[(F+G)\phi_1 - (F-G)\phi_2 - (F+G)\phi_3 + (F-G)\phi_4] + \frac{1}{\mu}Y \\
\xi_4 &= \frac{1}{4\beta}(\phi_1 - \phi_2 + \phi_3 - \phi_4) \\
\xi_5 &= \frac{1}{4\nu}(\phi_1 + \phi_2 - \phi_3 - \phi_4) \\
\xi_6 &= \frac{1}{4\nu}(\phi_1 - \phi_2 - \phi_3 + \phi_4) \\
\xi_7 &= \frac{1}{4\gamma}(\phi_1 + \phi_2 + \phi_3 + \phi_4)
\end{aligned}
\qquad (5\text{-}10)
$$

In the limit $\rho \to 0$, the normal coodinates are:

$$
\begin{aligned}
\xi_1 &= \frac{mr(r + r_o)}{\sqrt{I_T}}(\phi_1 + \phi_2 + \phi_3 + \phi_4) + \sqrt{I_T}\,\theta' \\
\xi_2 &= \frac{mr}{\sqrt{M+4m}}[-\phi_1 \sin\omega_o t - \phi_2 \cos\omega_o t + \phi_3 \sin\omega_o t + \phi_4 \cos\omega_o t] + X\sqrt{M+4m} \\
\xi_3 &= \frac{mr}{\sqrt{M+4m}}[\phi_1 \cos\omega_o t - \phi_2 \sin\omega_o t - \phi_3 \cos\omega_o t - \phi_4 \sin\omega_o t] + Y\sqrt{M+4m} \\
\xi_4 &= \frac{\sqrt{mr}}{2}[\phi_1 - \phi_2 + \phi_3 - \phi_4] \\
\xi_5 &= \frac{\sqrt{mr}}{2}\sqrt{\frac{M+2m}{M+4m}}[\phi_1 + \phi_2 - \phi_3 - \phi_4] \\
\xi_6 &= \frac{\sqrt{mr}}{2}\sqrt{\frac{M+2m}{M+4m}}[\phi_1 - \phi_2 - \phi_3 + \phi_4] \\
\xi_7 &= \frac{\sqrt{mr}}{2}\sqrt{\frac{I_o}{I_T}}[\phi_1 + \phi_2 + \phi_3 + \phi_4]
\end{aligned}
$$

5.3 Orthogonal Transformation of Potential Energy

$$[V] = \begin{pmatrix} \bar{p} & 0 & -q & 0 & 0 & 0 & 0 \\ 0 & \bar{p} & 0 & -q & 0 & 0 & 0 \\ -q & 0 & \bar{p} & 0 & 0 & 0 & 0 \\ 0 & -q & 0 & \bar{p} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{where } p = \left(mr + \rho\frac{r^2}{2}\right) r_o \omega_o^2$$

$$q = \left(mr + \rho\frac{r^2}{2}\right)^2 \omega_o^2 / \mathcal{M}$$

$$\bar{p} = p + q$$

$$[B]^T [V] [B] = [B]^T [V] \begin{pmatrix} 0 & 0 & 0 & \beta & \nu & \nu & \gamma \\ 0 & 0 & 0 & -\beta & \nu & -\nu & \gamma \\ 0 & 0 & 0 & \beta & -\nu & -\nu & \gamma \\ 0 & 0 & 0 & -\beta & -\nu & \nu & \gamma \\ \alpha & 0 & 0 & 0 & 0 & 0 & -\gamma\frac{4b}{c} \\ 0 & \mu & 0 & 0 & F\nu & -G\nu & 0 \\ 0 & 0 & \mu & 0 & -G\nu & -F\nu & 0 \end{pmatrix}$$

$$= [B]^T \begin{pmatrix} 0 & 0 & 0 & p\beta & \hat{p}\nu & \hat{p}\nu & p\gamma \\ 0 & 0 & 0 & p\beta & \hat{p}\nu & -\hat{p}\nu & p\gamma \\ 0 & 0 & 0 & p\beta & -\hat{p}\nu & -\hat{p}\nu & p\gamma \\ 0 & 0 & 0 & p\beta & -\hat{p}\nu & \hat{p}\nu & p\gamma \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4p\beta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4\hat{p}\nu^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4\hat{p}\nu^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4p\gamma^2 \end{pmatrix} \qquad \text{where } \hat{p} = p + 2q$$

Thus, in normal coordinates, potential energy is:

$$\frac{1}{2}\,(\xi_1 \cdots\cdots \xi_7)\,[B]^T\,[V]\,[B]\begin{bmatrix}\xi_1\\ \vdots\\ \vdots\\ \xi_7\end{bmatrix}$$

$$= \frac{1}{2}\,(\xi_1 \cdots\cdots \xi_7)\begin{pmatrix}\omega_1^2 &&&&&&\\ & \omega_2^2 &&&&&\\ && \omega_3^2 &&&&\\ &&& \omega_4^2 &&&\\ &&&& \omega_5^2 &&\\ &&&&& \omega_6^2 &\\ &&&&&& \omega_7^2\end{pmatrix}\begin{pmatrix}\xi_1\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \xi_7\end{pmatrix}$$

∴ potential energy $\boxed{\text{P. E.} = \frac{1}{2}\cdot\sum_{i=1}^{7}\omega_i^2\,\xi_i^2}$ in normal coordinates.

∴ the Lagrangian L in normal coordinates ξ_i is in the form:

$$\boxed{L = \frac{1}{2}\sum_{i=1}^{7}\left(\dot{\xi}_i^2 - \omega_i^2\,\xi_i^2\right)} \qquad (5\text{-}11)$$

where the normal frequencies ω_i are given by:

$\omega_1 = \omega_2 = \omega_3 = 0$ [Triple degeneracy: constant rotation, constant X translation, and constant Y translation]

$$\omega_4 = \sqrt{4\beta^2 p} = \sqrt{\frac{p}{a}}$$

$\lim_{\rho \to 0} \omega_4 = \omega_o\sqrt{\frac{r_o}{r}}$ [Uncoupled mode: angular momenta of booms cancel each other]

$$\omega_5 = \sqrt{4\nu^2 \hat{p}} = \sqrt{\frac{\hat{p}}{a} \Big/ \left(1 - \frac{2(mr + \rho\frac{r^2}{2})^2}{\mathcal{M} a}\right)}$$

$$\lim_{\rho \to o} \omega_5 = \omega_o \sqrt{\frac{r_o}{r}} \left(1 + \frac{m}{M} \frac{r + r_o}{r_o} + 0\left(\frac{m^2}{M^2} \frac{(r + r_o)^2}{r_o^2}\right)\right) \text{ for small } \frac{m}{M} \ll 1$$

$$\omega_6 = \omega_5$$

$$\omega_7 = \sqrt{4 p\gamma^2} \quad \sqrt{\frac{p}{a}} \sqrt{\frac{1}{1 - \frac{4b^2}{ac}}}$$

$$\lim_{\rho \to o} \omega_7 = \omega_o \sqrt{\frac{r_o}{r} \frac{I_T}{I_o}}$$

[Couples mode: total angular momenta of booms is nonzero]

These results are identical to those obtained in the previous chapter. There are three nonzero distinct frequencies as compared to two nonzero distinct frequencies in the case of no-translation formulation. However, the frequency of the modes with translation is very close to the uncoupled mode frequency. A general oscillation composing of various modes therefore exhibits beats phenomenon, because:

$$\Delta\omega = \omega_5 - \omega_4 = \omega_o \sqrt{\frac{r_o}{r}} \left(1 + \frac{m}{M} \frac{r + r_o}{r_o} + \ldots - 1\right)$$

$$= \omega_o \sqrt{\frac{r_o}{r}} \frac{m}{M} \frac{r + r_o}{r_o} \qquad (5\text{-}12)$$

which gives a beat period ΔT of the order of over one thousand seconds for 1975. This phenomenon would be absent in a formulation without translation.

CHAPTER 6

OUT-OF-PLANE OSCILLATIONS (OPO)

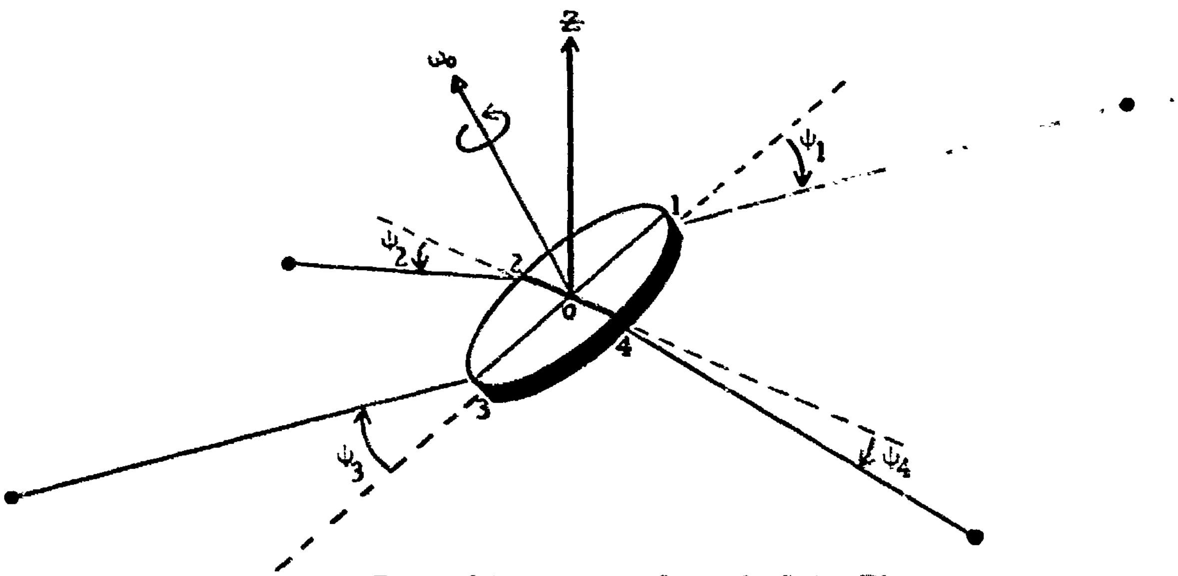
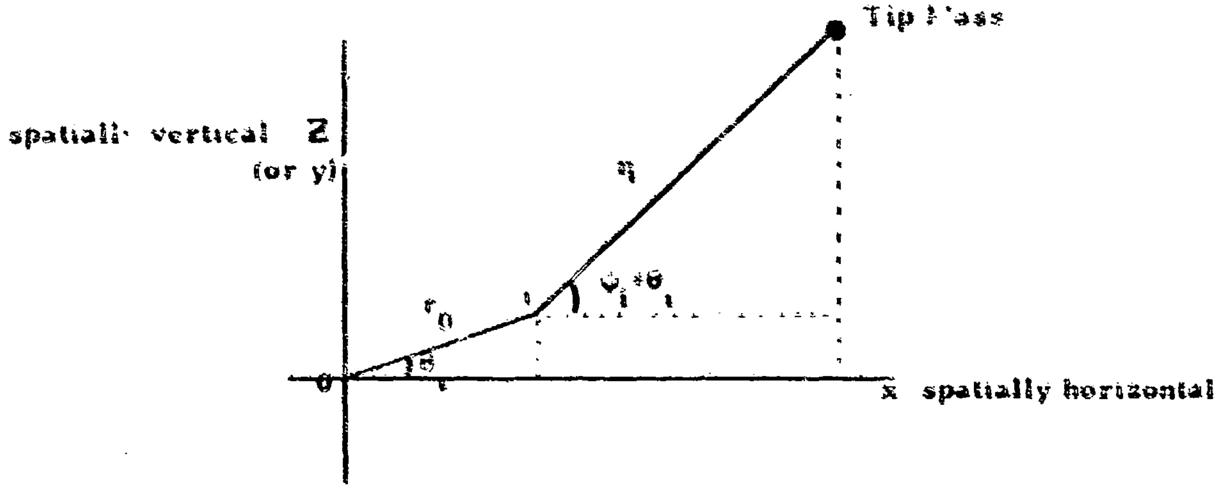


Figure 4. Boom Movements Out-of -Spin-Plane

Notation:

0 = center of hub
1, 2, 3, 4 = exit points of booms
Z = spatially fixed Z-axis
OPO axis of hub are 103 and 204.
θ_i are angles subtended by io with the horizontal plane
ψ_i are angles subtended by i-th boom with the corresponding hub axis
θ_i and ψ_i are defined positive if counterclockwise from equilibrium, as viewed at 4 or 1.
x_i and y_i are measured in a corotating frame about Z-axis; x_i is the length of horizontal projection of a tip mass from 0 on the horizontal plane.



6.1 Coordinates of Tip Masses

$$\left.\begin{aligned} x_i &= r_i \cos(\theta_i + \psi_i) + r_0 \cos\theta_i \\ y_i &= r_i \sin(\theta_i + \psi_i) + r_0 \sin\theta_1 \end{aligned}\right\} i = 1, 2$$

$$\left.\begin{aligned} x_i &= r_i \cos(\theta_i + \psi_i + \pi) + r_0 \cos(\theta_i + \pi) \\ y_i &= r_i \sin(\theta_i + \psi_i + \pi) + r_0 \sin(\theta_i + \pi) \end{aligned}\right\} i = 3, 4$$

$$\left.\begin{aligned} \dot{x}_i &= \dot{r}_i \cos(\theta_i + \psi_i) - (\dot\theta_i + \dot\psi_i)\, r_i \sin(\theta_i + \psi_i) - \dot\theta_i\, r_0 \sin\theta_i \\ \dot{y}_i &= \dot{r}_i \sin(\theta_i + \psi_i) + (\dot\theta_i + \dot\psi_i)\, r_i \cos(\theta_i + \psi_i) + \dot\theta_i\, r_0 \cos\theta_i \end{aligned}\right\} i = 1, 2$$

$$\left.\begin{aligned} \dot{x}_i &= -\dot{r}_i \cos(\theta_{i-2} + \psi_i) + (\dot\theta_{i-2} + \dot\psi_i)\, r_i \sin(\theta_{i-2} + \psi_i) + \dot\theta_{i-2}\, r_0 \sin\theta_{i-2} \\ \dot{y}_i &= -\dot{r}_i \sin(\theta_{i-2} + \psi_i) - (\dot\theta_{i-2} + \dot\psi_i)\, r_i \cos(\theta_{i-2} + \psi_i) - \dot\theta_{i-2}\, r_0 \cos\theta_{i-2} \end{aligned}\right\} i = 3, 4$$

$$\begin{aligned} \sum_{i=1}^{2} (\dot{x}_i^2 + (\dot{y}_i + \dot{Z})^2) = \sum_{i=1}^{2} [&\dot{r}_i^2 + (\dot\theta_i + \dot\psi_i)^2 r_i^2 + \dot\theta_i^2 r_0^2 + \dot{Z}^2 + 2\dot{r}_i \dot\theta_i r_0 \\ &\sin\psi_i + 2\dot\theta_i (\dot\theta_i + \dot\psi_i)\, r_i r_0 \cos\psi_i + 2\dot{Z}\, (\dot{r}_i \sin(\theta_i + \psi_i) \\ &+ (\dot\theta_i + \dot\psi_i)\, r_i \cos(\theta_i + \psi_i) + \dot\theta_i\, r_0 \cos\theta_i)] \end{aligned}$$

$$\begin{aligned} \sum_{i=3}^{4} (\dot{x}_i^2 + (\dot{y}_i + \dot{Z})^2) = \sum_{i=3}^{4} [&\dot{r}_i^2 + (\dot\theta_{i-2} + \dot\psi_i)^2 r_i^2 + \dot\theta_{i-2}^2\, r_0^2 + \dot{Z}^2 \\ &+ 2\dot{r}_i \dot\theta_{i-2}\, r_0 \sin\psi_i + 2\dot\theta_{i-2} (\dot\theta_{i-2} + \dot\psi_i)\, r_i r_0 \cos\psi_i \\ &- 2\dot{Z}\, (\dot{r}_i \sin(\theta_{i-2} + \psi_i) + (\dot\theta_{i-2} + \dot\psi_i)\, r_i \cos(\theta_{i-2} + \psi_i) + \\ &\dot\theta_{i-2}\, r_0 \cos\theta_{i-2})] \end{aligned}$$

Kinetic Energy K. E. of the system due to OPO =

$$K.E. = 1/2\, I_{103}\, \dot\theta_1^{\,2} + 1/2\, I_{204}\, \dot\theta_2^{\,2} + 1/2 \sum_{i=1}^{4} m_i\, (\dot x_i^{\,2} + (\dot y_i + \dot Z)^2 + \frac{M}{2} \cdot \dot Z^2$$

where I_{103} is moment of inertia of hub about 103 axis.

I_{204} is about 204 axis.

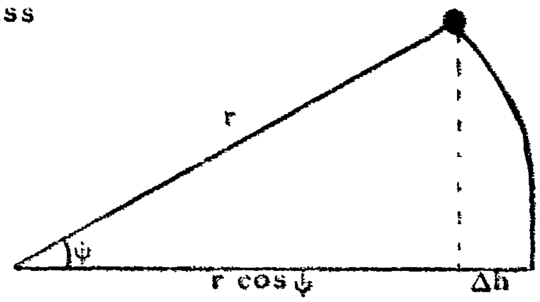
y_i is the y-coordinate of i-th tip mass without translation.

Z is translation of hub, (Z and y_i are in the same direction)

Potential Energy P. E. of a tip mass

$= mg\Delta h$

$= m\, \omega_0^2\ (r + r_0)\, r\, (1 - \cos\psi)$



Lagrangian L of the system due to OPO is

$L = K.E. - P.E.$

$$L = 1/2\, I_{103}\, \dot\theta_1^{\,2} + 1/2\, I_{204}\, \dot\theta_2^{\,2} + \frac{M}{2}\, \dot Z^2$$

$$+ 1/2\, m \sum_{i=1}^{2} [\dot x_i^{\,2} + (\dot y_i + \dot z)^2] + 1/2 m \sum_{i=3}^{4} [\dot x_i^{\,2} + (\dot y_i + \dot z)^2]$$

$$- \sum_{i=1}^{4} m\, \omega_0^{\,2}\, (r_i + r_0)\, r_i\, (1 - \cos\psi_i)$$

$$L = 1/2\, I_1\, \dot\theta_1^{\,2} + 1/2\, I_2\, \dot\theta_2^{\,2} + 1/2\, m \sum_{i=1}^{4} \{\dot r_i^{\,2} + (\dot\theta_i + \dot\psi_i)^2\, r_i^{\,2} +$$

$$+ \dot\theta_i^{\,2}\, r_0^{\,2} + \dot Z^2 + 2\, \dot\theta_i\, r_0\, (\dot r_i \sin\psi_i + (\dot\theta_i + \dot\psi_i)\, r_i \cos\psi_i)\}$$

$$+ m \dot Z \sum_{i=1}^{2} [\dot r_i \sin(\theta_i + \psi_i) + (\dot\theta_i + \dot\psi_i)\, r_i \cos(\theta_i + \psi_i)]$$

$$- m \dot Z \sum_{i=3}^{4} [\dot r_i \sin(\theta_i + \psi_i) + (\dot\theta_i + \dot\psi_i)\, r_i \cos(\theta_i + \psi_i)]$$

$$- m\, \omega_0^{\,2} \sum_{i=1}^{4} (r_i + r_0)\, r_i\, (1 - \cos\psi_i) - 1/2\, s \sum_{i=1}^{4} \dot\psi_i^{\,2} + \frac{M}{2}\, \dot Z^2$$

where $\theta_3 = \theta_1$

$\theta_4 = \theta_2$

s = spring constant

6.2 Out-of-Plane Lagrangian

Including deployment/retraction, mass density of wire, spring constant, and translation, the Lagrangian L for OPO is:

$$
\begin{aligned}
L = {} & 1/2\, I_1 \dot{\theta}_1^2 + 1/2\, I_2 \dot{\theta}_2^2 \\
& + 1/2 \sum_{i=1}^{4} \Big\{ (m+\rho r_i)\dot{r}_i^2 + (m r_i^2 + \rho \frac{r_i^3}{3}) (\dot{\theta}_i + \dot{\psi}_i)^2 \\
& + (m + \rho r_i) (\dot{\theta}_i^2 r_0^2 + Z^2) + 2\dot{\theta}_1 r_0 \quad [(m + \rho r_i) \dot{r}_i \sin\psi_i \\
& + (\dot{\theta}_i + \dot{\psi}_i) (m r_i + \rho \frac{r_i^2}{2}) \cos \psi_i] \Big\} \\
& + \dot{Z} \sum_{i=1}^{2} \Big\{ (m + \rho r_i)\dot{r}_i \sin (\theta_i + \psi_i) + (m r_i + \rho \frac{r_i^2}{2}) (\dot{\theta}_i + \dot{\psi}_i) . \cos (\theta_i + \psi_i) \Big\} \\
& - \dot{Z} \sum_{i=3}^{4} \Big\{ (m + \rho r_i) r_i \sin (\theta_{i-2} + \psi_i) + (m r_i + \rho \frac{r_i^2}{2}) (\dot{\theta}_{i-2} + \dot{\psi}_i) \cos \\
& \qquad (\theta_{i-2} + \psi_i) \Big\} \\
& + \frac{M}{2} Z^2 - \omega_0^2 \sum_{i=1}^{4} \Big\{ | (m r_i^2 + \rho \frac{r_i^3}{3}) + r_0 (m r_i + \rho \frac{r_i^2}{2}) | . (1 - \cos \psi_i) \Big\} \\
& \qquad - 1/2\, s \sum_{i=1}^{4} \psi_i^2
\end{aligned}
\qquad (6\text{-}1)
$$

where $\theta_3 = \theta_1$ and $\theta_4 = \theta_2$

6.3 Harmonic Approximation

We now simplify the Lagrangian L for the purpose of finding normal frequencies, modes and coordinates. We assume:

1) $\dot{r} = 0$; no deployment/retraction

2) $r_i = r$; all booms are of equal length

3) $\rho = 0$; no boom wire mass density

4) $s = 0$; no spring constant (s is small compared to centrifugal force)

5) $I_1 = I_2$ because of $r_i = r$

6) ψ_i, θ_i, and z are of the order $O(\epsilon)$; only terms up to $O(\epsilon^2)$ are kept in harmonic approximation. Hence, $\cos\psi_i = 1 - \psi_i^2/2 + \ldots$

Assumptions 3) and 4) are for the sake of simplicity in algebra. If we are willing to tolerate some slight complexity in algebra, we can relax the restrictions 3) and 4), even though they are not essential for the analysis of normal modes.

Thus, the Lagrangian L simplifies to:

$$L = 1/2\, I_1 (\dot{\theta}_1^2 + \dot{\theta}_2^2) + 1/2 \sum_{i=1}^{4} \left\{ (m\, r_i^2 + \rho \frac{r_i^3}{3})(\dot{\theta}_i + \dot{\psi}_i)^2 + (m + \rho r_i) (\dot{\theta}_i^2 r_0^2 + \dot{z}^2) + 2\,\dot{\theta}_i r_0 (\dot{\theta}_i + \dot{\psi}_i)(m\, r_i + \rho \frac{r_i^2}{2})(1 - \frac{\psi_i^2}{2}) \right\} + \dot{Z} \sum_{i=1}^{2} \left\{ (\dot{\theta}_i + \dot{\psi}_i)(m\, r_i + \rho \frac{r_i^2}{2})(1 - \frac{(\theta_i + \psi_i)^2}{2}) \right\} - \dot{Z} \sum_{i=3}^{4} \left\{ (\dot{\theta}_{i-2} + \dot{\psi}_i)(m\, r_i + \rho \frac{r_i^2}{2}) [1 - \frac{(\theta_{i-2} + \psi_i)^2}{2}] \right\} - 1/2\, \omega_0^2 \sum_{i=1}^{4} \left\{ [(m\, r_i^2 + \rho \frac{r_i^3}{3}) + (m\, r_i + \rho \frac{r_i^2}{2})\, r_0]\, \psi_i^2 \right\} + \frac{M}{2} \dot{z}^2 - \frac{s}{2} \sum_{i=1}^{4} \psi_i^2$$

Since $I_{1T} = I_1 + 2\,[(m\, r^2 + \rho \frac{r^3}{3}) + (m + \rho r)\, r_0^2 + 2\,(m\, r + \rho \frac{r^2}{2})\, r_0]$

is a constant in case of no deployment/retraction, we further simplify the Lagrangian L to the following:

$$L = 1/2\, I_{1T} (\dot{\theta}_i^2 + \dot{\theta}_2^2) + 1/2\, m\, \dot{Z}^2$$ (cont. next page)

$$+ 1/2 \sum_{i=1}^{4} \left\{ (m\, r^2 + \rho \frac{r^3}{3}) \; (2\, \dot{\theta}_i\, \dot{\psi}_i + \dot{\psi}_i^2) + (m + \rho r)\, \dot{Z}^2 \right.$$

$$\left. + 2\, (m\, r + \rho \frac{r^2}{2})\, r_0\, \dot{\theta}_i\, \dot{\psi}_i \right\}$$

$$+ \dot{Z}\, (m\, r + \rho \frac{r^2}{2}) \left\{ + \dot{\psi}_1 + \dot{\psi}_2 - \dot{\psi}_3 - \dot{\psi}_4 \right\}_2$$

$$- 1/2\, [\, \omega_0^2\, (m\, r^2 + \rho \frac{r^3}{3} + r_0\, (m\, r + \rho \frac{r^2}{2}\,)) + s]\, \sum_{i=1}^{4} \psi_i^2 \qquad (6\text{-}2)$$

In matrix notation, the Lagrangian in harmonic approximation is:

$$L = 1/2\, (\dot{\psi}_1\, \dot{\psi}_2\, \dot{\psi}_3\, \dot{\psi}_4\, \dot{\theta}_1\, \dot{\theta}_2\, \dot{Z}) \begin{pmatrix} a & 0 & 0 & 0 & b & 0 & +d \\ 0 & a & 0 & 0 & 0 & b & +d \\ 0 & 0 & a & 0 & b & 0 & -d \\ 0 & 0 & 0 & a & 0 & b & -d \\ b & 0 & b & 0 & c & 0 & 0 \\ 0 & b & 0 & b & 0 & c & 0 \\ +d & +d & -d & -d & 0 & 0 & \mathcal{M} \end{pmatrix} \begin{pmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \\ \dot{\psi}_3 \\ \dot{\psi}_4 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{Z} \end{pmatrix}$$

$$- 1/2\, (\psi_1\, \psi_2\, \psi_3\, \psi_4\, \theta_1\, \theta_2\, Z) \begin{pmatrix} q & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \theta_1 \\ \theta_2 \\ Z \end{pmatrix}$$

where $a = m\, r^2 + \rho \frac{r^3}{3}$

$$b = m\, r^2 + \rho \frac{r^3}{3} + r_0\, (m\, r + \rho \frac{r^2}{2}\,) = a + r_0\, d$$

$$c = I_{1T}$$

$$d = m\, r + \rho \frac{r^2}{2}$$

$$\mathcal{M} = M + 4\, (m + \rho\, r) \simeq M + 4m$$

$$q = \omega_0^2\, [m\, r^2 + \rho \frac{r^3}{3} + r_0\, (m\, r + \rho \frac{r^2}{2}\,)] + s$$

The Lagrangian has been written in such a form that the matrices [T] and [V] are manifested.

$$L = 1/2\,(\dot{\psi}_1 \ldots \dot{Z})\;[T]\;\begin{bmatrix}\dot{\psi}_1\\ \vdots \\ \dot{Z}\end{bmatrix} - 1/2\,(\psi_1 \ldots Z)\;[V]\;\begin{bmatrix}\psi_1\\ \vdots \\ Z\end{bmatrix}$$

where:

$$[T] = \begin{pmatrix} \alpha & 0 & 0 & 0 & b & 0 & +d \\ 0 & a & 0 & 0 & 0 & b & +d \\ 0 & 0 & a & 0 & b & 0 & -d \\ 0 & 0 & 0 & a & 0 & b & -d \\ b & 0 & b & 0 & c & 0 & 0 \\ 0 & b & 0 & b & 0 & c & 0 \\ +d & +d & -d & -d & 0 & 0 & \mathcal{M} \end{pmatrix}$$

$$[V] = \begin{pmatrix} q & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad (6-3)$$

From symmetry considerations, or other methods, the orthogonal matrix has the form:

$$[B] = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & +1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -\alpha & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\alpha & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & \beta \end{pmatrix} \qquad (6-4)$$

corresponding to the seven modes: pure translation, pure hub rotations, coupled oscillations, saddle mode, and jelly-fish mode, respectively.

The unknown quantities α and β can be found by considering conservations of angular and linear mementa. For the coupled mode (4th or 5th column of B), the sum of angular momenta of booms and hub is (see Appendix B)

$$I_{1T}\,\dot{\theta} + 2\left[m\,r^2 + \rho\frac{r^3}{3} + r_0\left(m\,r + \rho\frac{r^2}{2}\right)\right]\dot{\psi} = 0$$

so that

$$\dot{\theta} = -\frac{2b}{I_{1T}}\;\dot{\psi}$$

Thus, for $\dot{\psi}_1 = \dot{\psi}_3 = 1$, we find $\alpha = -\frac{2b}{c}$

For the jelly-fish mode (7th column of B), conservation of linear momentum gives:

$$M\,\dot{Z} = \sum_{i=1}^{4} \left[\left(m\, r_i + \rho \frac{r_i^2}{2}\right) \dot{\psi}_i \cos \psi_i - (m + \rho r_i)\, \dot{Z} \right]$$

so that:

$$\dot{Z} = \frac{4\left(m\, r + \rho \frac{r^2}{2}\right) \dot{\psi}}{M + 4m}$$

Thus, we find $\beta = \dfrac{4\left(m\, r + \rho \frac{r^2}{2}\right)}{M + 4m} = \dfrac{4d}{\mathcal{M}}$

So now, we have an orthogonal matrix B:

$$[B] = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -2b/c & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2b/c & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & \frac{4d}{\mathcal{M}} \end{pmatrix} \qquad (6\text{-}5)$$

where $[B]$ is orthogonal in the unusual sense: $[B]^T\,[T]\,[B] = I$

$$[B]^T\,[T]\,[B] = [B]^T \begin{pmatrix} d & 0 & b & a-\frac{2b^2}{c} & 0 & -a & -a+\frac{4d^2}{\mathcal{M}} \\ d & b & 0 & 0 & a-\frac{2b^2}{c} & a & -a+\frac{4d^2}{\mathcal{M}} \\ -d & 0 & b & a-\frac{2b^2}{c} & 0 & a & a-\frac{4d^2}{\mathcal{M}} \\ -d & b & 0 & 0 & a-\frac{2b^2}{c} & -a & a-\frac{4d^2}{\mathcal{M}} \\ 0 & 0 & c & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & 0 & 0 & 0 \\ \mathcal{M} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$
= \begin{bmatrix}
\mathcal{M} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & c & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2(a-\frac{2b^2}{c}) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2(a-\frac{2b^2}{c}) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4a & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4(a-\frac{4d^2}{\mathcal{M}})
\end{bmatrix}
$$

which is diagonal but not unity because B has not yet been normalized.

Normalized Orthogonal Matrix [B] is:

$$
[B] = \begin{bmatrix}
0 & 0 & 0 & n_3 & 0 & -n_4 & -n_5 \\
0 & 0 & 0 & 0 & n_3 & n_4 & -n_5 \\
0 & 0 & 0 & n_3 & 0 & n_4 & n_5 \\
0 & 0 & 0 & 0 & n_3 & -n_4 & n_5 \\
0 & 0 & n_2 & -\frac{2b}{c}\, n_3 & 0 & 0 & 0 \\
0 & n_2 & 0 & 0 & -\frac{2b}{c}\, n_3 & 0 & 0 \\
n_1 & 0 & 0 & 0 & 0 & 0 & \frac{4d}{\mathcal{M}}\, n_5
\end{bmatrix}
$$

where $n_1 = \dfrac{1}{\sqrt{\mathcal{M}}}$

$$n_2 = \frac{1}{\sqrt{I_{1T}}}$$

$$n_3 = \frac{1}{\sqrt{2(a-b^2/c)}}$$

$$n_4 = \frac{1}{\sqrt{a}}$$

$$n_5 = \frac{1}{2\sqrt{a-\frac{4d^2}{\mathcal{M}}}}$$

(a, b, c, d, $\mathcal{M}$ are defined on Page 69)

This matrix [B] satisfies: $[B]^T\,[T]\,[B] = I$

Now we can define a new set of coordinates $\{\xi_i\}$ related to the original coordinates $\{\psi_i, \ldots, z\}$ by the following equation:

$$\begin{pmatrix} \psi_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ z \end{pmatrix} = [B] \begin{pmatrix} \xi_8 \\ \xi_9 \\ \xi_{10} \\ \cdot \\ \cdot \\ \xi_{14} \end{pmatrix} \qquad (6\text{-}6)$$

where $\{\xi_i\}$ are the normal coordinates.

The kinetic energy in terms of these normal coordinates is:

$$K.E. = 1/2\ (\dot{\xi})^T\ (B)^T\ (T)\ (B)\ (\dot{\xi})$$

But, the unusual orthogonality relation is:

$$[B]^T\ [T]\ [B] = I$$

Hence, the kinetic energy in normal coordinates becomes:

$$\boxed{K.E. = 1/2 \sum_{i=8}^{14} \dot{\xi}_i\ \dot{\xi}_i} \qquad (6\text{-}7)$$

where: $[\xi] = [B]^{-1}\ [\psi]$

The inverse $[B]^{-1}$ is calculated in Appendix G. It is:

$$[B]^{-1} = \begin{bmatrix} \frac{d}{\mathcal{M} n_1} & \frac{d}{\mathcal{M} n_1} & -\frac{d}{\mathcal{M} n_1} & -\frac{d}{\mathcal{M} n_1} & 0 & 0 & \frac{1}{n_1} \\ 0 & \frac{b}{cn_2} & 0 & \frac{b}{cn_2} & 0 & \frac{1}{n_2} & 0 \\ \frac{b}{cn_2} & 0 & \frac{b}{cn_2} & 0 & \frac{1}{n_2} & 0 & 0 \\ \frac{1}{2n_3} & 0 & \frac{1}{2n_3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2n_3} & 0 & \frac{1}{2n_3} & 0 & 0 & 0 \\ -\frac{1}{4n_4} & \frac{1}{4n_4} & \frac{1}{4n_4} & -\frac{1}{4n_4} & 0 & 0 & 0 \\ -\frac{1}{4n_5} & -\frac{1}{4n_5} & \frac{1}{4n_5} & \frac{1}{4n_5} & 0 & 0 & 0 \end{bmatrix}$$

where a, b, c, d, $\mathcal{M}$, n_1, ... n_5 are defined on pages 70 and 73.

$$\lim_{\rho \to 0} \frac{d}{\mathcal{M} n_1} = \frac{mr}{\sqrt{M + 4m}}, \qquad \lim_{\rho \to 0} \frac{1}{2n_3} = \sqrt{\frac{mr^2}{2} \frac{I_{10}}{I_{1T}}}$$

$$\lim_{\rho \to 0} \frac{b}{cn_2} = \frac{mr(r + r_0)}{\sqrt{I_{1T}}}, \qquad \lim_{\rho \to 0} \frac{1}{4n_4} = \frac{\sqrt{m}\, r}{2}$$

$$\lim_{\rho \to 0} \frac{1}{n_1} = \sqrt{M + 4m}, \qquad \lim_{\rho \to 0} \frac{1}{4n_5} = \frac{\sqrt{m}\, r}{2} \sqrt{\frac{M}{M + 4m}}$$

$$\lim_{\rho \to 0} \frac{1}{n_2} = \sqrt{I_{1T}}, \qquad \lim_{\rho \to 0} I_{1T} = I_{10} + 2\, m\, (r + r_0)^2$$

Therefore, the normal coordinates for out-of-plane oscillations are:

$$\boxed{\begin{aligned}
\xi_8 &= \frac{d}{\mathcal{M} n_1} (\psi_1 + \psi_2 - \psi_3 - \psi_4) + \frac{1}{n_1} Z \\
\xi_9 &= \frac{b}{cn_2} (\psi_2 + \psi_4) + \frac{1}{n_2} \theta_1 \\
\xi_{10} &= \frac{b}{cn_2} (\psi_1 + \psi_3) + \frac{1}{n_2} \theta_2 \\
\xi_{11} &= \frac{1}{2n_3} (\psi_1 + \psi_3) \\
\xi_{12} &= \frac{1}{2n_3} (\psi_2 + \psi_4) \\
\xi_{13} &= \frac{1}{4n_4} (-\psi_1 + \psi_2 + \psi_3 - \psi_4) \\
\xi_{14} &= \frac{1}{4n_5} (-\psi_1 - \psi_2 + \psi_3 + \psi_4)
\end{aligned}} \qquad (6\text{-}8)$$

In the Limit $\rho \to 0$, the normal coordinates are:

$$\xi_8 = \frac{mr}{\sqrt{M + 4m}} (\psi_1 + \psi_2 - \psi_3 - \psi_4) + \sqrt{M + 4m} \quad Z$$

$$\xi_9 = \frac{mr\,(r + r_0)}{\sqrt{I_{1T}}} (\psi_2 + \psi_4) + \sqrt{I_{1T}} \quad \theta_1$$

$$\xi_{10} = \frac{mr\,(r + r_0)}{\sqrt{I_{1T}}} (\psi_1 + \psi_3) + \sqrt{I_{1T}} \quad \theta_2$$

$$\xi_{11} = \sqrt{\frac{mr^2}{2} \frac{I_{10}}{I_{1T}}} (\psi_1 + \psi_3)$$

$$\xi_{12} = \sqrt{\frac{mr^2}{2} \frac{I_{10}}{I_{1T}}} (\psi_2 + \psi_4)$$

$$\xi_{13} = \frac{\sqrt{m}\,r}{2} (-\psi_1 + \psi_2 + \psi_3 - \psi_4)$$

$$\xi_{14} = \frac{\sqrt{m}\,r}{2} \sqrt{\frac{M}{M + 4m}} (-\psi_1 - \psi_2 + \psi_3 + \psi_4)$$

6.4 Potential Energy Orthogonal Transformation

$$[V] = \begin{pmatrix} q & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where $q = \omega_0^2 \; [mr^2 + \rho\frac{r^3}{3} + r_0\,(mr + \rho\frac{r^2}{2})] + s$

$$[B]^T [V] [B] = [B]^T [V] \begin{bmatrix} 0 & 0 & 0 & n_3 & 0 & -n_4 & -n_5 \\ 0 & 0 & 0 & 0 & n_3 & n_4 & -n_5 \\ 0 & 0 & 0 & n_3 & 0 & n_4 & n_5 \\ 0 & 0 & 0 & 0 & n_3 & -n_4 & n_5 \\ 0 & 0 & n_2 & -\frac{2b}{c} n_3 & 0 & 0 & 0 \\ 0 & n_2 & 0 & 0 & -\frac{2b}{c} n_3 & 0 & 0 \\ n_1 & 0 & 0 & 0 & 0 & 0 & \frac{4d}{\mathcal{M}} n_5 \end{bmatrix}$$

$$= [B]^T \begin{bmatrix} 0 & 0 & 0 & qn_3 & 0 & -qn_4 & -qn_5 \\ 0 & 0 & 0 & 0 & qn_3 & qn_4 & -qn_5 \\ 0 & 0 & 0 & qn_3 & 0 & qn_4 & qn_5 \\ 0 & 0 & 0 & 0 & qn_3 & -qn_4 & qn_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2qn_3^{\,2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2qn_3^{\,2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4qn_4^{\,2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4qn_5^{\,2} \end{bmatrix}$$

Thus, in normal coordinates, the potential energy is:

$$P.E. = 1/2\,(\xi_8 \ldots \xi_{14})\,(B)^T\,(V)\,(B) \begin{bmatrix} \xi_8 \\ \vdots \\ \xi_{14} \end{bmatrix}$$

$$= 1/2\,(\xi_8 \ldots \xi_{14}) \begin{bmatrix} \omega_8^2 & & & & \\ & \omega_9^2 & & & \\ & & \omega_{10}^2 & & \\ & & & \ddots & \\ & & & & \omega_{14}^2 \end{bmatrix} \begin{bmatrix} \xi_8 \\ \vdots \\ \xi_{14} \end{bmatrix}$$

$$\boxed{P.E. = 1/2 \sum_{i=8}^{14} \omega_i^2 \ \xi_i^2}$$ in normal coordinates.

$\therefore$ the Lagrangian L in normal coordinates ξ_i is in the form:

$$\boxed{L = 1/2 \sum_{i=8}^{14} (\dot{\xi}_i^2 - \omega_i^2 \ \xi_i^2)} \qquad (6\text{-}9)$$

where

the normal frequencies ω_i are given as follows:

$\omega_8 = \omega_9 = \omega_{10} = 0$ [Constant translation, and constant rotations]

Coupled Modes

$$\omega_{11} = \sqrt{2q}\, n_3 = \sqrt{\frac{q}{a - \frac{2b^2}{c}}} = \sqrt{\frac{\omega_0^2\,(a + r_0 d) + s}{a - \frac{2b^2}{c}}}$$

$$\omega_{12} = \omega_{11}$$

Uncoupled Saddle Mode

$$\omega_{13} = \sqrt{\frac{q}{a}}$$

Jelly-Fish Mode

$$\omega_{14} \ \sqrt{\frac{q}{a - \frac{4d^2}{\pi}}}$$

Limit $\rho \to 0$

$$\lim_{\rho \to 0} a = m\, r^2$$

$$\lim_{\rho \to 0} b = m\, r\,(r + r_0)$$

$$\lim_{\rho \to 0} I_{1T} = I_{10} + 2m\,(r + r_0)^2$$

$$\lim_{\rho \to 0} d = mr$$

$$\lim_{\rho \to 0} q = \omega_0^2\, mr\,(r + r_0) + s$$

For s small compared to $\omega_0^2\, mr\,(r + r_0)$, $q \doteq \omega_0^2\, mr\,(r + r_0)$.

Thus, the out-of-plane natural frequencies in this limit and harmonic approximation are:

$$\omega_8 = \omega_9 = \omega_{10} = 0$$

$$\left.\begin{aligned} \omega_{11} &= \omega_0 \sqrt{\frac{r + r_0}{r} \quad \frac{I_{1T}}{I_{10}}} \\ \omega_{12} &= \omega_{11} \end{aligned}\right\} \quad \text{Coupled Mode}$$

$$\left.\omega_{13} = \omega_0 \sqrt{\frac{r + r_0}{r}}\right\} \quad \text{Uncoupled Saddle Mode}$$

$$\left.\omega_{14} = \omega_0 \sqrt{\frac{r + r_0}{r} \quad \frac{M + 4m}{M}}\right\} \quad \text{Jelly-Fish Mode}$$

CHAPTER 7

UNEQUAL LENGTH BOOMS

7.1 Boom Pairs of Unequal Length, Normal Modes without Translation

New oscillation frequencies and modes emerge if the boom pairs become unequal in length, after deployment/retraction. For large ratio of hub to tip masses, good approximation can be achieved for a formulation without tr ation. As discussed in earlier chapters, there are usually two approaches to solve an oscillational dynamics problem analytically: (1) evaluate the secular determinant to find eigenvalues and eigenvectors; (2) write down the mode matrices from symmetry considerations and perform orthonormal transformations.

Because of the asymmetry of the system for this case, it is reasonable to follow approach (1) first, i.e., to find the natural modes. The second approach will be attempted later with translation included.

In harmonic approximation, with no damping, the total Lagrangian L of the system without translation is:

$$L = L_0 + \sum_{i=1}^{4} L_i$$

where

$$L_0 = 1/2\, I_0\, \dot{\theta}^2$$

$$L_i = 1/2\,(mr_i^2 + \rho\frac{r_i^3}{3})\,\dot{\phi}_i^2 + \dot{\theta}[(mr_i^2 + \rho\frac{r_i^3}{3})\,\dot{\phi}_i + (mr_i + \rho\frac{r_i^2}{2})\,r_0 \cos\phi_i\,\dot{\phi}_i] + 1/2\,\dot{\theta}^2\,[(mr_i^2 + \rho\frac{r_i^3}{3}) + (m + \rho r_i)\,r_0^2 + 2(mr_i + \rho\frac{r_i^2}{2})\,r_0 \cos\phi_i] \qquad (7\text{-}1)$$

where $r_1 = r_3 = r$, and $r_2 = r_4 = r'$ (let $r' > r$). Thus, ignoring damping terms, the Lagrangian L in terms of the five generalized coordinates ϕ_1, ϕ_2, ϕ_3, ϕ_4, θ', where $\theta' = \theta(t) - \theta(0)$, can be written as:

$$L = 1/2\,(\dot{\phi}_1 \ldots \dot{\theta}')\,(T)\begin{pmatrix}\dot{\phi}_1\\ \vdots \\ \dot{\theta}'\end{pmatrix} - 1/2\,(\phi_1 \ldots \theta')\,(V)\begin{pmatrix}\phi_1\\ \vdots \\ \theta'\end{pmatrix} \qquad (7\text{-}2)$$

where

$$[T] = \begin{bmatrix} a & 0 & 0 & 0 & b \\ 0 & a' & 0 & 0 & b' \\ 0 & 0 & a & 0 & b \\ 0 & 0 & 0 & a' & b' \\ b & b' & b & b' & c \end{bmatrix} \qquad [V] = \begin{bmatrix} p & 0 & 0 & 0 & 0 \\ 0 & p' & 0 & 0 & 0 \\ 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & p' & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{7-3}$$

where $a = mr^2 + \rho\frac{r^3}{3}$, $\quad a' = mr'^2 + \rho\frac{r'^3}{3}$

$$b = mr^2 + \rho\frac{r^3}{3} + r_0\,(mr + \rho\frac{r^2}{2}), \qquad b' = mr'^2 + \rho\frac{r^3}{3} + r_0\,(mr' + \rho\frac{r^2}{2})$$

$$c = I_T\,(\phi = 0) = I_0 + \sum_{i=1}^{4} [mr_i^2 + \rho\frac{r_i^3}{3} + (m + \rho r_i)\,r_0^2$$

$$+ 2\,(mr_i + \rho\frac{r_i^2}{2})\,r_0]$$

$$p = (mr + \rho\frac{r^2}{2})\,r_0\ \ \omega_0^2, \qquad p' = (mr' + \rho\frac{r'^2}{2})\,r_0\ \ \omega_0^2$$

[Let $r' > r$, so that $a' > a$, $b' > b$, $p' > p$]

The Lagrangian L leads to the following five equations of motion:

$$\sum_{j=1}^{5} T_{ij}\,\ddot{\phi}_j + \sum_{j=1}^{5} V_{ij}\,\phi_j = 0 \quad (i=1,\ldots 5) \qquad \text{where } \phi_5 \text{ denotes } \theta'.$$

For oscillatory motion near the equilibrium, the solutions $\phi_i = \phi_i\,(t=0)$ $\exp(-i\omega t)$ can be tried, leading to the following simultaneous equations:

$$\sum_{j=1}^{5} (V_{ij}\,\phi_j - \omega^2\,T_{ij}\,\phi_j) = 0$$

Nontrivial solutions exist if the vanishing condition of the secular determinant is satisfied.

$$\det \begin{vmatrix} a\omega^2 - p & 0 & 0 & 0 & b\omega^2 \\ 0 & a'\omega^2 - p' & 0 & 0 & b'\omega^2 \\ 0 & 0 & a\omega^2 - p & 0 & b\omega^2 \\ 0 & 0 & 0 & a'\omega^2 - p' & b'\omega^2 \\ b\omega^2 & b'\omega^2 & b\omega^2 & b'\omega^2 & c\omega^2 \end{vmatrix} = 0 \tag{7-4}$$

which can be evaluated by using Laplace's expansion. The result is a fifth order equation in ω^2:

$$\omega^2 (a\omega^2 - p)(a'\omega^2 - p')\ [c(a\omega^2 - p)(a'\omega^2 - p') -$$

$$2b^2\omega^2(a'\omega^2 - p') - 2b'^2\omega^2(a\omega^2 - p)] = 0 \tag{7-5}$$

Thus, the eigenfrequencies are:

$$\omega_1 = 0$$

$$\omega_2 = \sqrt{p/a}$$

$$\omega_3 = \sqrt{p'/a'}$$

ω_4 and ω_5 are not quite as simple as the others, but they satisfy a quadratic equation in ω^2:

$$\frac{c}{2\omega^2} = \frac{b^2}{a\omega^2 - p} + \frac{b'^2}{a'\omega^2 - p'} \tag{7-6}$$

with the solutions:

$$\left.\begin{matrix}\omega_4\\ \omega_5\end{matrix}\right\} = \frac{-B \pm [B^2 - 4App'c]^{1/2}}{2A} \tag{7-7}$$

where $A = c\,aa' - 2(b^2 a' + b'^2 a)$

$B = 2(b^2 p' + b'^2 p) - c(pa' + p'a)$

In the lim.ι ρ→0, the eigenfrequencies are:

$$\omega_1 = 0$$

$$\omega_2 = \sqrt{\frac{r_0}{r}}\quad \omega_0$$

$$\omega_3 = \sqrt{\frac{r_0}{r'}}\quad \omega_0$$

$$\lim_{r'\to r} \omega_4 = \sqrt{\frac{r_o}{r}}\quad \omega_0$$

$$\lim_{r'\to r} \omega_5\ \sqrt{\frac{r_o}{r}\frac{L_r}{I_o}}\quad \omega_0$$

The modes corresponding to each eigenfrequency ω_i are determined by the equations of motion:

$$(a\omega^2 - p)\,\phi_1 + b\omega^2\,\theta' = 0$$

$$(a'\omega^2 - p')\,\phi_2 + b'\omega^2\,\theta' = 0$$

$$(a\omega^2 - p)\,\phi_3 + b\omega^2\,\theta' = 0 \tag{7-7}$$

$$(a'\omega^2 - p')\,\phi_4 + b'\omega^2\,\theta' = 0$$

$$b(\phi_1 + \phi_3) + b'(\phi_2 + \phi_4) + c\theta' = 0$$

Mode 1: $\omega = \omega_1 = 0,\ \theta' = \phi_i = 0\ (i = 1, \ldots 4)$ which implies no oscillation.

Mode 2: $\omega = \omega_2 = \sqrt{p/a}\ ,\ \phi_1 = -\phi_3\ ,\ \phi_2 = \phi_4 = 0,\ \theta' = 0$

Mode 3: $\omega = \omega_3 = \sqrt{p'/a'}\ ,\ \phi_1 = \phi_3 = 0,\ \phi_2 = -\phi_4,\ \theta' = 0$

These two modes (2 and 3) are uncoupled modes because hub rotation is unaffected. Their frequencies are not new, and for finite hub mass, there is translation involved.

Modes 4 and 5: These two are coupled modes. Equation (7-7) gives:

$$\phi_1 = \phi_3 = -\frac{b\omega^2}{a\omega^2 - p}\ \theta' \tag{7-8}$$

$$\phi_2 = \phi_4 = -\frac{b'\omega^2}{a'\omega^2 - p'}\ \theta$$

where ω^2 satisfies equation (7-6) for these modes. Therefore,

$$\frac{\phi_1}{\phi_2} = \frac{\phi_3}{\phi_4} = \frac{b\,a'}{b'\,a}\ \frac{\omega^2 - \omega_3^2}{\omega^2 - \omega_2^2} \tag{7-9}$$

There are two roots of equation (7-6). Without working out the details of the explicit solution, we can analyze their behavior by plotting together the L. H. S. and the R. H. S. of the equation.

$$\text{L.H.S.} = \frac{c}{2\omega^2}$$

$$\text{R.H.S.} = \frac{b^2 / a}{\omega^2 - \omega_2^2} + \frac{b'^2 / a'}{\omega^2 - \omega_3^2} \qquad (7\text{-}10)$$

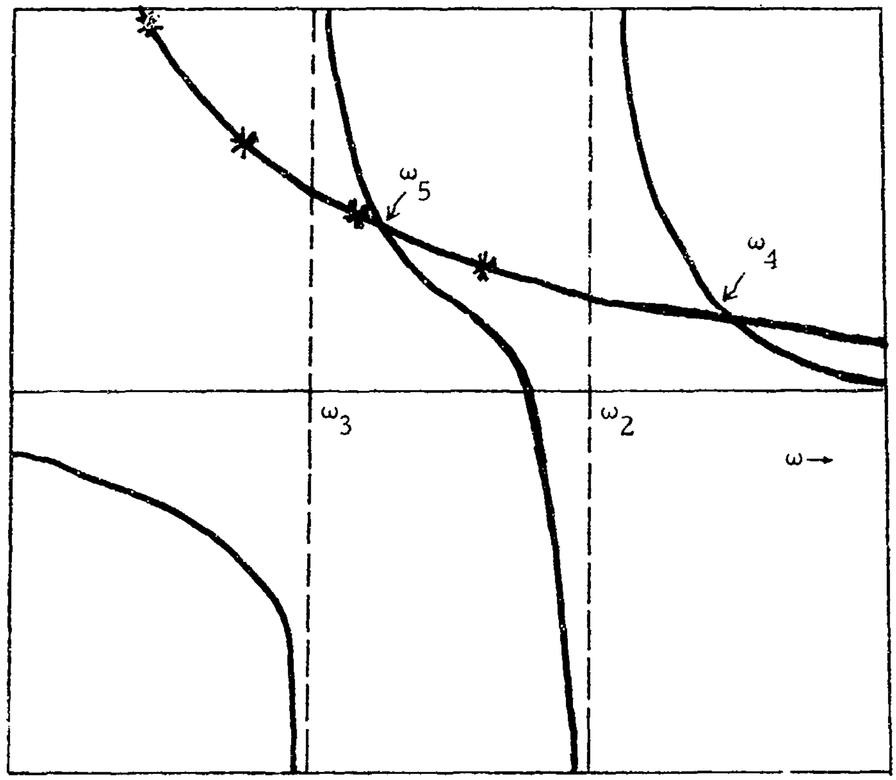


Figure 5. Locations of Modes 4 and 5 Frequencies
L. H. S. curve is denoted by *

From Figure 5, one observes that the higher coupled frequency ω^4 is higher than both uncoupled frequencies ω_2 and ω_3, while the lower coupled frequency ω_5 is intermediate between ω_2 and ω_3.

For mode 4, $\omega_4 > \omega_2 > \omega_3$, so that equations (7-8) and (7-9) give the mode pattern: $(\phi, \phi', \phi, \phi', -\theta')$, i.e., all booms move in the same direction with the same frequency in phase pairwise, while the hub moves in opposite direction.

For mode 5, $\omega_2 > \omega_5 > \omega_3$; equations (7-8) and (7-9) give the mode pattern: $(\phi, -\phi', \phi, -\phi', \theta')$, i.e., booms adjacent to each other are completely out of phase, and therefore oscillate with the same frequency despite unequal lengths, while the hub is in phase with the short pair of booms.

For summary, we display the schematic diagrams of the modes below.

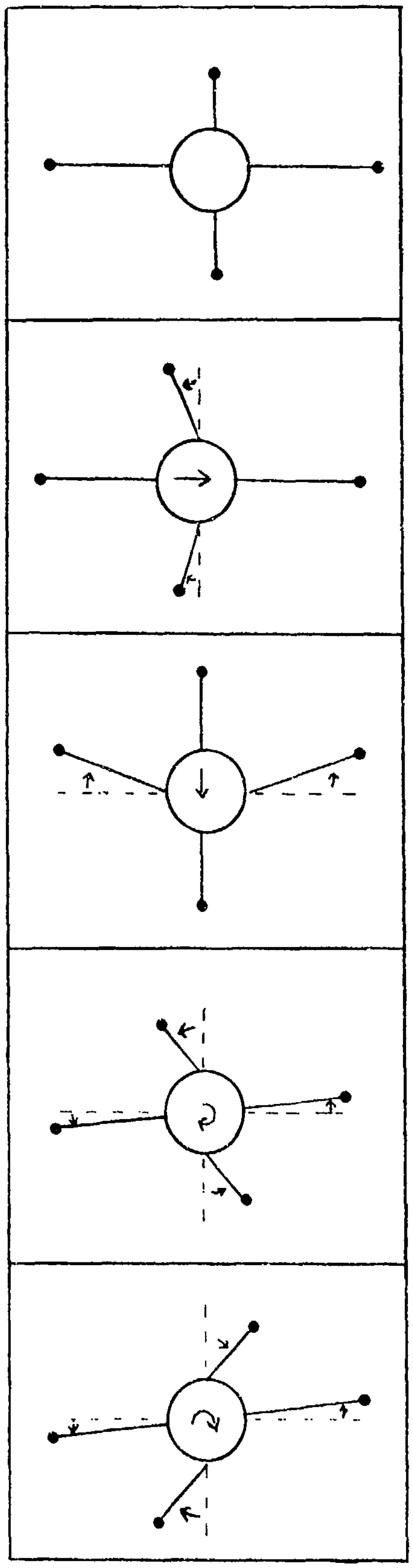


Figure 6. Mode Shapes for Unequal Length Boom Pairs

7.2 Four Booms All of Different Lengths

The normal modes of a heavy rotating hub with all four booms of different lengths can be found easily by using the secular determinant method, in the "no translation" approximation, the error introduced by neglecting translation is of the order of:

$$\sum_{i=1}^{4} m_i/(M_0 + \sum_{i=1}^{4} m_i)$$

as discussed in Chapter 4. For satellite 1975, this ratio is less than 1.6%.

The method given in Section 7.1 yields a secular determinant condition for the case of all different boom lengths:

$$\det \begin{vmatrix} a_1\omega^2 - p_1 & 0 & 0 & 0 & b_1\omega^2 \\ 0 & a_2\omega^2 - p_2 & 0 & 0 & b_2\omega^2 \\ 0 & 0 & a_3\omega^2 - p_3 & 0 & b_3\omega^2 \\ 0 & 0 & 0 & a_4\omega^2 - p_4 & b_4\omega^2 \\ b_1\omega^2 & b_2\omega^2 & b_3\omega^2 & b_4\omega^2 & c\,\omega^2 \end{vmatrix} = 0 \qquad (7\text{-}11)$$

where $a_i = mr_i^2 + \rho\dfrac{r_i^3}{3}$

$$b_i = mr_i^2 + \rho\frac{r_i^3}{3} + r_0\left(mr_i + \rho\frac{r_i^2}{2}\right)$$

$$c = I_T(\phi = 0) = I_0 + \sum_{i=1}^{4}\Big[mr_i^2 + \rho\frac{r_i^3}{3} + (m + \rho r_i)\, r_0^2$$

$$+ 2\left(mr_i + \rho\frac{r_i^2}{2}\right) r_0\Big]$$

$$p_i = \left(mr_i + \rho\frac{r_i^2}{2}\right) r_0\ \omega_0^2$$

[Let $r_1 < r_2 < r_3 < r_4$ so that the uncoupled frequencies ω_i satisfy:

$$\omega_1 > \omega_2 > \omega_3 > \omega_4$$

where:

$$\omega_i = \sqrt{r_0/r_i}\ \omega_0$$

which is the uncoupled frequency.]

The 5x5 secular determinant can be evaluated without difficulty using Laplace's expansion. The result can be further simplified by dividing throughout by: $\omega^2 \prod_{i=1}^{4} (a_i \omega^2 - p_i)$, to the form:

$$\frac{c}{\omega^2} = \sum_{i=1}^{4} \frac{b_i^2 / a_i}{\omega^2 - \omega_i^2} \tag{7-12}$$

Instead of seeking explicit solutions for this 4th order polynomial equation in ω^2, it is most illuminating to plot the L.H.S. and R.H.S. of the equation against ω^2, in order to understand the properties of the roots.

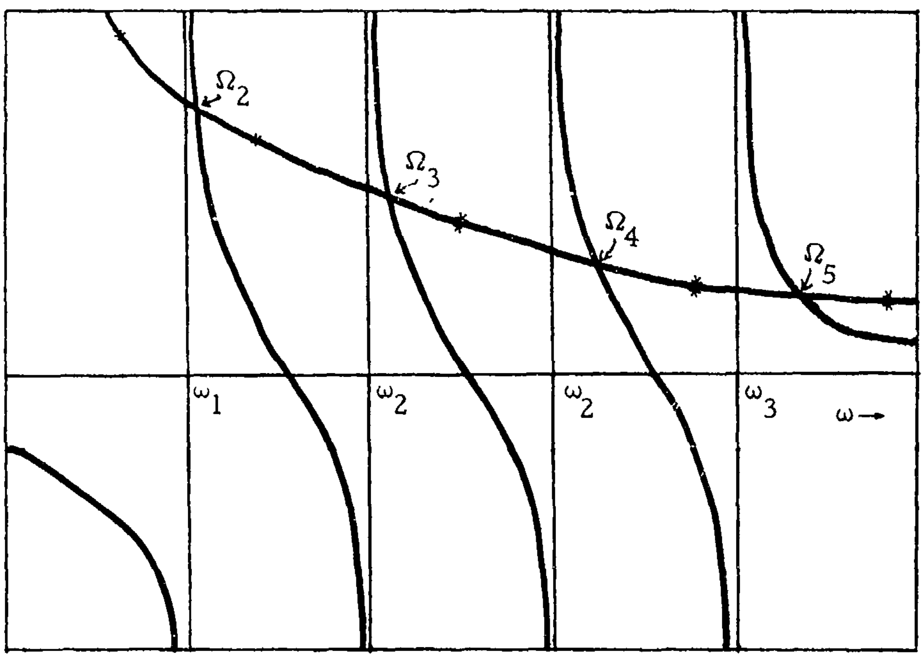


Figure 7. Frequency Location of Modes 2, 3, 4 and 5 for a heavy hub with four booms of different lengths. L.H.S. of Equ. (7-12) is denoted by * curve. [The trivial root $\omega = 0$ of Equ. (7-11) is not shown.]

It should be pointed out that all four modes are coupled modes, with slight translation due to unbalanced boom lengths. All Ω_i are higher than uncoupled ω_i, respectively. In case the hub becomes infinitely heavy $\lim_{C \to \infty} \Omega_i \to \omega_i$ assymptotically.

The mode shapes can be found by using Equ. (7-7) for each Ω_i. The schematic diagrams of the normal modes of a heavy hub with four booms of different lengths are displayed below:

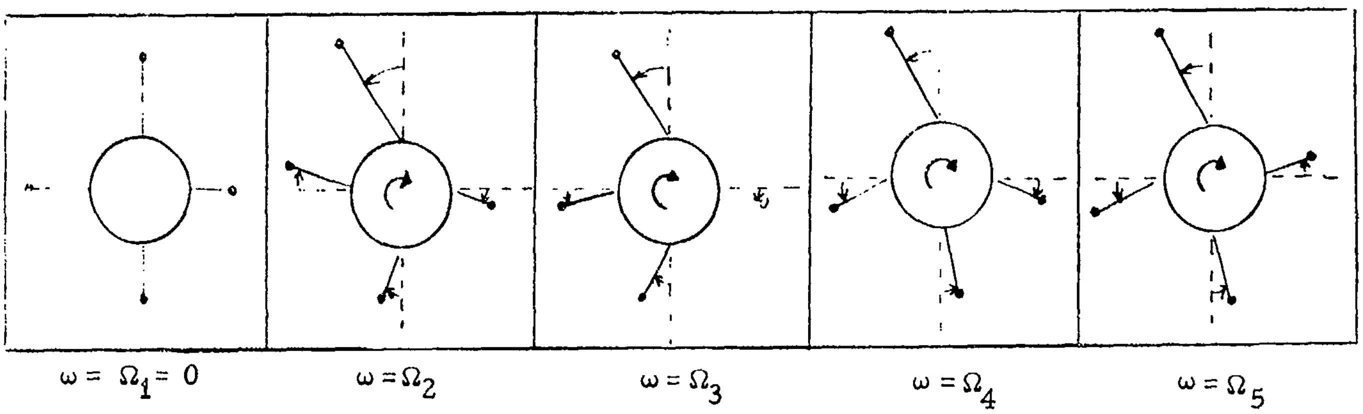


Figure 8. Mode Shapes of a heavy hub with booms all of different lengths.

Note that for oscillations in a pure mode, all booms and hub oscillate with the same frequency, not withstanding the unequal lengths of the booms.

7.3 Unequal Boom Pair Lengths, with Translation

In matrix notatation, the Lagrangian in harmonic approximation as given in Chapter 5 is generalized for the case of unequal boom lengths to:

$$L = \frac{1}{2}\,(\dot\phi_1\ \dot\phi_2\ \dot\phi_3\ \dot\phi_4\ \dot\theta'\ \dot X'\ \dot Y')\begin{pmatrix} a & 0 & 0 & 0 & b & -d & e \\ 0 & a_1 & 0 & 0 & b_1 & -e_1 & -d_1 \\ 0 & 0 & a & 0 & b & d & -e \\ 0 & 0 & 0 & a_1 & b_1 & e_1 & d_1 \\ b & b_1 & b & b_1 & c & 0 & 0 \\ -d & -e_1 & d & e_1 & 0 & \mathfrak{M} & 0 \\ e & -d_1 & -e & d_1 & 0 & 0 & \mathfrak{M} \end{pmatrix}\begin{pmatrix} \dot\phi_1 \\ \dot\phi_2 \\ \dot\phi_3 \\ \dot\phi_4 \\ \dot\theta' \\ \dot X' \\ \dot Y' \end{pmatrix}$$

$$-\ \frac{1}{2}\,(\phi_1\ \phi_2\ \phi_3\ \phi_4\ \theta'\ X'\ Y')\begin{pmatrix} \bar p & 0 & -q & 0 & 0 & 0 & 0 \\ 0 & \bar p_1 & 0 & -q & 0 & 0 & 0 \\ -q & 0 & \bar p & 0 & 0 & 0 & 0 \\ 0 & -q & 0 & \bar p_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \theta' \\ X' \\ Y' \end{pmatrix}$$

where
$$a = mr^2 + \rho r^3/3$$
$$b = (mr^2 + \rho r^3/3) + r_o\,(mr + \rho r^2/2)$$
$$c = I_T\,(\phi_i = 0)$$
$$d = (mr + \rho r^2/2)\sin\omega_o t \qquad\qquad q = 2\,(mr + \rho r^2/2)^2\,\omega_o^2/\mathfrak{M}$$
$$e = (mr + \rho r^2/2)\cos\omega_o t \qquad\qquad \bar p = p + q$$
$$\mathfrak{M} = M + 4m$$
$$p = (mr + \rho r^2/2)\,r_o\,\omega_o^2$$

and the subscript 1 stands for functions of r_1 in place of r ($r \neq r_1$).

From symmetry considerations, with guiding light shed from the preceding sections, the orthogonal matrix [B] can be written down:

$$[\,B\,] = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & \mu_- & \mu_+ \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & \mu_- & \mu_+ \\ 0 & 0 & 0 & -1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & \nu_+ & \nu_- \\ 0 & 1 & 0 & f_1 & -g & 0 & 0 \\ 0 & 0 & 1 & g_1 & f & 0 & 0 \end{bmatrix} \qquad (7\text{-}13)$$

corresponding to the seven modes: pure rotation with no boom oscillation, pure translations with no boom oscillation, uncoupled modes with translations, and coupled modes without translation.

The unknown matrix elements (f, g, etc.) can be determined by considering conservation of linear and angular momenta. For the first translation mode (4th column of B), linear momentum conservation gives:

$$\mathcal{M} f_1 = -\sum_{i=1}^{4} (mr_i + \rho r_i^2/2) \sin(\theta + \phi_i + \delta_i)\, \dot\phi_i$$

$$\mathcal{M} g_1 = -\sum_{i=1}^{4} (mr_i + \rho r_i^2/2) \cos(\theta + \phi_i + \delta_i)\, \dot\phi_i$$

with $\dot\phi_1 = \dot\phi_3 = 0$ and $\dot\phi_2 = -\dot\phi_4 = 1$. Thus, we find

$$f = 2e/\mathcal{M}, \qquad f_1 = 2e_1/\mathcal{M}$$

$$g = 2d/\mathcal{M}, \qquad g_1 = 2d_1/\mathcal{M}$$

where

$$d = (mr + \rho \frac{r^2}{2}) \sin \omega_o t$$

$$e = (mr + \rho \frac{r^2}{2}) \cos \omega_o t$$

$$d_1 = d(r \to r_1)$$

$$e_1 = e(r \to r_1)$$

For the coupled modes (last two columns of B), the sum of angular momenta of booms and hub is:

$$b(\dot{\phi}_1 + \dot{\phi}_3) + b_1(\dot{\phi}_2 + \dot{\phi}_4) + c\dot{\theta} = 0$$

so that

$$2(b\mu + b_1) + c\nu = 0 \qquad (7 - 14)$$

where μ is the ratio of amplitudes for booms (1, 3) to booms (2, 4). The unknowns μ, ν are also linked by the first four equation of Equ. (7-7), via ω^2.

$$\frac{p_1}{a_1 + b_1\nu} = \omega^2 = \frac{p\mu}{a\mu + b\nu} \qquad (7 - 15)$$

Equations (7-14) and (7-15) together yield quadratic equations in μ and ν :

$$\mu^2 + \left[\frac{b_1}{b} - \frac{bp_1}{b_1p} + c\left(\frac{ap_1 - a_1p}{2bb_1p}\right)\right]\mu - \frac{p_1}{p} = 0 \qquad (7 - 16)$$

$$\nu^2 + \left[\left(\frac{b_1}{b} + \frac{bp_1}{b_1p}\right)\frac{2b}{c} + \left(\frac{a_1p - p_1a}{b_1p}\right)\right]\nu + \frac{2}{c}\,\frac{a_1p - p_1a}{p} = 0 \qquad (7\text{-}17)$$

It follows from Equ. (7-16) that, in particular, the product of roots of μ is:

$$\mu_+ \mu_- = -\frac{p_1}{p} \qquad (7\text{-}18)$$

Substituting this result into Equ. (7-15) yields:

$$\mu_+ (a\mu_- + b\nu) = -(a_1 + b_1\nu) \qquad (7\text{-}19)$$

Explicit solutions of μ and ν (Equ. 7-16, 17) are given by:

$$\mu_\pm = \frac{1}{2}\left[-B \pm \sqrt{B^2 + \frac{4p_1}{p}}\right]$$

$$\nu_\pm = \frac{1}{2}\left[-B_\nu \pm \sqrt{B_\nu^2 - 8(a_1p - p_1a)/pc}\right] \qquad (7\text{-}20)$$

where $$B = \left(\frac{b_1}{b} - \frac{bp_1}{b_1p}\right) + \frac{c}{2b}\left(\frac{ap_1 - a_1p}{b_1p}\right)$$

$$B_\nu = \left(\frac{b_1}{b} + \frac{bp_1}{b_1p}\right)\frac{2b}{c} + \left(\frac{a_1p - p_1a}{b_1p}\right)$$

In the limit $r_1 \to r$, we have $a_1 \to a$, etc, so that , for equal boom lengths,

$$\lim_{r_1 \to r} \mu_\pm = \pm 1$$

$$\lim_{r_1 \to r} \nu_\pm = \begin{cases} 0 \\ -4b/c \end{cases}$$

But limiting values of the mode variables for equal boom length case are (Chapter 3 and 4)

$$(-1,\ 1,\ -1,\ 1,\ 0,\ 0,\ 0)$$

and $$(\ 1,\ 1,\ 1,\ 1,\ -4b/c,\ 0,\ 0)$$

Thus, the correct combinations of μ and ν for unequal boom length cases are identified:

$$(\mu_-,\ 1,\ \mu_-,\ 1,\ \nu_+,\ 0,\ 0)$$

and $$(\mu_+,\ 1,\ \mu_+,\ 1,\ \nu_-,\ 0,\ 0)$$

where $\mu_\pm$ and $\nu_\pm$ are given in Equ. (7-20)

7.4 Diagonalization of T-matrix

The orthogonal matrix [B] should satisfy the unusual orthogonality relation:

$$[\,B\,]^T [\,T\,][\,B\,] = I \qquad (7\text{-}21)$$

If [B] is not normalized, then the unity matrix I is replaced by a diagonal matrix.

$$[\,B\,]^T[\,T\,][\,B\,] = [\,B\,]^T \begin{pmatrix} a & 0 & 0 & 0 & b & -d & e \\ 0 & a_1 & 0 & 0 & b_1 & -e_1 & -d_1 \\ 0 & 0 & a & 0 & b & d & -e \\ 0 & 0 & 0 & a_1 & b_1 & e_1 & d_1 \\ b & b_1 & b & b_1 & c & 0 & 0 \\ -d & -e_1 & d & e_1 & 0 & \hbar & 0 \\ e & -d_1 & -e & d_1 & 0 & 0 & \hbar \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & \mu & \mu_+ \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & \mu_- & \mu_+ \\ 0 & 0 & 0 & -1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & \nu_+ & \nu_- \\ 0 & 1 & 0 & f_1 & -g & 0 & 0 \\ 0 & 0 & 1 & g_1 & f & 0 & 0 \end{pmatrix}$$

$$= [\,B\,]^T \begin{pmatrix} b & -d & e & 0 & -(a-dg-ef) & a\mu_- + b\nu_+ & a\mu_+ + b\nu_- \\ b_1 & -e_1 & -d_1 & (a_1 - e_1 f_1 - d_1 g_1) & 0 & a_1 + b_1\nu_+ & a_1 + b_1\nu_- \\ b & d & -e & 0 & (a-dg-ef) & a\mu_- + b\nu_+ & a\mu_+ + b\nu_- \\ b_1 & e_1 & d_1 & (-a + ef_1 + d_1 g_1) & 0 & a_1 + b_1\nu_+ & a_1 + b_1\nu_- \\ c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \hbar & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \hbar & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$
\begin{bmatrix}
c & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \hbar & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \hbar & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2(a_1-e_1f_1-d_1g_1) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2(a-ef-dg) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2(a\mu_-^2+b\mu_-\nu_+ +a_1+b_1\nu_+) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2(a\mu_+^2+b\mu_+\nu_- +a_1+b_1\nu_-)
\end{bmatrix}
$$

which is diagonal but not unity because [B] has not yet been normalized. The normalized orthogonal matrix [B] is:

$$
[\,B\,] = \begin{bmatrix}
0 & 0 & 0 & 0 & -n_5 & \mu_- n_6 & \mu_+ n_7 \\
0 & 0 & 0 & n_4 & 0 & n_6 & n_7 \\
0 & 0 & 0 & 0 & n_5 & \mu_- n_6 & \mu_+ n_7 \\
0 & 0 & 0 & -n_4 & 0 & n_6 & n_7 \\
n_1 & 0 & 0 & 0 & 0 & \nu_+ n_6 & \nu_- n_7 \\
0 & n_2 & 0 & f_1 n_4 & -g n_5 & 0 & 0 \\
0 & 0 & n_3 & g_1 n_4 & f n_5 & 0 & 0
\end{bmatrix} \tag{7-22}
$$

where

$$n_1 = 1/\sqrt{I_T}$$

$$n_2 = n_3 = 1/\sqrt{\hbar}$$

$$n_4 = 1/[\,2\,(a_1 - e_1 f_1 - d_1 g_1)]^{1/2}$$

$$n_5 = 1/[\,2\,(a - ef - dg)\,]^{1/2}$$

$$n_6 = 1/[\,2\,(\mu_-(a\mu_- + b\nu_+) + a_1 + b_1\nu_+)\,]^{1/2}$$

$$n_7 = 1/[\,2\,(\mu_+(a\mu_+ + b\nu_-) + a_1 + b_1\nu_-)\,]^{1/2}$$

where the symbols a_1, e_1, etc. are defined on page 90.

It is readily verified that this orthonormal matrix satisfies Equ. (7-19).

Now we can define a new set of coordinates $\{\xi_i\}$ related to the original coordinates $\{\phi_1, \ldots, Y'\}$ by the following equation:

$$\begin{bmatrix} \phi_1 \\ \cdot \\ \cdot \\ \cdot \\ Y' \end{bmatrix} = [\,B\,] \begin{bmatrix} \xi_1 \\ \cdot \\ \cdot \\ \cdot \\ \xi_7 \end{bmatrix} \tag{7-23}$$

where ξ_i are the normal coordinates. The kinetic energy T in terms of these normal coordinates is:

$$\text{K.E.} = \frac{1}{2}[\,\dot{\xi}\,]^T [\,B\,]^T [\,T\,][\,B\,][\,\dot{\xi}\,]$$

$$\text{K.E.} = \frac{1}{2}\sum_{i=1}^{7} \dot{\xi}_i \; \dot{\xi}_i \tag{7-24}$$

The inverse $[\,B]^{-1}$ is calculated in Appendix H. It is

$$[\,B\,]^{-1} = \begin{bmatrix} (\nu_+-\nu_-)/\zeta n & (\mu_-\nu_- - \nu_+\mu_+)/\zeta n_1 & (\nu_+-\nu_-)/\zeta n_1 & (\mu_-\nu_- - \mu_+\nu_+)/\zeta n_1 & n_1^{-1} & 0 & 0 \\ -g/2n_2 & -f_1/2n_2 & g/2n_2 & f_1/2n_2 & 0 & n_2^{-1} & 0 \\ f/2n_3 & -g_1/2n_3 & -f/2n_3 & g_1/2n_3 & 0 & 0 & n_3^{-1} \\ 0 & 1/2n_4 & 0 & -1/2n_4 & 0 & 0 & 0 \\ -1/2n_5 & 0 & 1/2n_5 & 0 & 0 & 0 & 0 \\ -1/\zeta n_6 & \mu_+/\zeta n_6 & -1/\zeta n_6 & \mu_+/\zeta n_6 & 0 & 0 & 0 \\ 1/\zeta n_7 & -\mu_-/\zeta n_7 & 1/\zeta n_7 & -\mu/\zeta n_7 & 0 & 0 & 0 \end{bmatrix}$$

where $\zeta = 2(\mu_+ - \mu_-)$

7.5 Orthogonal Transformation of V-Matrix

The orthogonal transformation of the potential energy V-matrix by the same B matrix determined in the preceding section gives:

$$[B]^T [V] [B] =$$

$$[B]^T \begin{pmatrix} \bar{p} & 0 & -q & 0 & 0 & 0 & 0 \\ 0 & \bar{p}_1 & 0 & -q_1 & 0 & 0 & 0 \\ -q & 0 & \bar{p} & 0 & 0 & 0 & 0 \\ 0 & -q_1 & 0 & \bar{p}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & -n_5 & \mu_- n_6 & \mu_+ n_7 \\ 0 & 0 & 0 & n_4 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & n_5 & \mu_- n_6 & \mu_+ n_7 \\ 0 & 0 & 0 & -n_4 & 0 & 1 & 1 \\ n_1 & 0 & 0 & 0 & 0 & \nu_+ n_6 & \nu_- n_7 \\ 0 & n_2 & 0 & f_1 n_4 & -g n_5 & 0 & 0 \\ 0 & 0 & n_3 & g_1 n_4 & f n_5 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & n_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & n_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & n_3 \\ 0 & n_4 & 0 & -n_4 & 0 & f_1 n_4 & g_1 n_4 \\ -n_5 & 0 & n_5 & 0 & 0 & -g n_5 & f n_5 \\ \mu_- n_6 & 1 & \mu_- n_6 & 1 & \nu_+ n_6 & 0 & 0 \\ \mu_+ n_7 & 1 & \mu_+ n_7 & 1 & \nu_- n_7 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & -\hat{p} n_5 & p\mu_- n_6 & p\mu_+ n_7 \\ 0 & 0 & 0 & \hat{p}_1 n_4 & 0 & p_1 n_6 & p_1 n_7 \\ 0 & 0 & 0 & 0 & \hat{p} n_5 & p\mu_- n_6 & p\mu_+ n_7 \\ 0 & 0 & 0 & -\hat{p}_1 n_4 & 0 & p_1 n_6 & p_1 n_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where $\hat{p} = \bar{p} + q = p + 2q$

$\hat{p}_1 = \bar{p}_1 + q_1 = p_1 + 2q_1$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\hat{p}_1 n_4{}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\hat{p} n_5{}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(p\mu_-^2 + p_1)n_6{}^2 & 2(p\mu_-\mu_+ + p_1)n_6 n_7 \\ 0 & 0 & 0 & 0 & 0 & 2(p\mu_-\mu_+ + p_1)n_6 n_7 & 2(p\mu_+^2 + p_1)n_7{}^2 \end{pmatrix}$$

The matrix elements (6, 7) and (7, 6) are zero because of Equ. (7-18). Matrix element (6, 6) can be written as:

$$\frac{p\mu_-^2 + p_1}{\mu_-(a\mu_- + b\nu_+) + a_1 + b_1\nu_+} = \frac{p_1}{a_1 + b_1\nu_+} \quad \text{or} \quad \frac{p\mu_-}{a\mu_- + b\nu_-}$$

by virtue of Equ. (7-15). Similarly, matrix element (7, 7) can be written as:

$$\frac{p_1}{a_1 + b_1\nu_-} \quad \text{or} \quad \frac{p\mu_+}{a\mu_+ + b\nu_-}$$

where $\nu_\pm$ and $\mu_\pm$ are solutions of the quadratic equation (7-17), and are given in Equ. (7-20).

In normal coordinates, the potential energy is:

$$\text{P. E.} = \frac{1}{2}(\xi_1 \cdots \xi_7) \begin{pmatrix} \omega_1^2 & & \\ & \ddots & \\ & & \omega_7^2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_7 \end{pmatrix}$$

As a result of the orthogonal transformation, the Lagrangian L becomes formally simple and elegant:

$$L = \frac{1}{2} \sum_{i=1}^{7} \left(\dot{\xi}_i^{\,2} - \omega_i^{\,2} \xi_i^{\,2} \right) \tag{7-25}$$

where
$$\omega_1 = 0$$
$$\omega_2 = 0$$
$$\omega_3 = 0$$
$$\omega_4 = \sqrt{2\hat{p}_1} \quad n_4 = \sqrt{\frac{\hat{p}_1/a_1}{1-(e_1f_1 + d_1g_1)/a_1}}$$
$$\omega_5 = \sqrt{2\hat{p}} \quad n_5 = \sqrt{\frac{\hat{p}/a}{1-(ef + dg)/a}}$$
$$\omega_6 = \sqrt{\frac{p_1}{a_1 + b_1\nu_+}} \quad \text{or} \quad \sqrt{\frac{p\mu_-}{a\mu_- + b\nu_+}}$$
$$\omega_7 = \sqrt{\frac{p}{a + b\nu_-}} \quad \text{or} \quad \sqrt{\frac{p\mu_+}{a\mu_+ + b\nu_-}}$$

where a, b, p, etc. have been defined in Secion 7.3, and $\nu_\pm$ are given in equ. (7-20). Their limiting values for vanishing wire mass are as follows:

$$\lim_{\rho\to 0} a = mr^2, \qquad \lim_{\rho\to 0} a_1 = mr_1^{\,2}$$

$$\lim_{\rho\to 0} b = mr(r + r_o), \qquad \lim_{\rho\to} b_1 = mr_1(r_1 + r_o)$$

$$\lim_{\rho\to 0} p = mrr_o\,\omega_o^{\,2} \qquad \lim_{\rho\to 0} p_1 = mr_1\,r_o\,\omega_o^{\,2}$$

$$\lim_{\rho\to 0} d = mr \sin \omega_o t, \qquad \lim_{\rho\to 0} d_1 = mr_1 \sin \omega_o t$$

$$\lim_{\rho\to 0} e = mr \cos \omega_o t, \qquad \lim_{\rho\to 0} e_1 = mr_1 \cos \omega_o t$$

$$\lim_{\rho\to 0} f = 2(m/\mathcal{M})\, r \cos\omega_o t, \qquad \lim_{\rho\to 0} f_1 = 2(m/\mathcal{M})\, r_1 \cos\omega_o t$$

$$\lim_{\rho\to 0} g = 2(m/\mathcal{M})\, r \sin\omega_o t, \qquad \lim_{\rho\to 0} g_1 = 2(m/\mathcal{M})\, r_1 \sin\omega_o t$$

$$\lim_{\rho\to 0} \nu_\pm = \frac{1}{2}\left[-\lim_{\rho\to 0} B_\nu \pm \sqrt{\lim_{\rho\to 0} B_\nu^{\,2} - 8mr_1\,(r_1 - r)/I_T}\,\right]$$

$$\lim_{\rho\to 0} B_\nu = \frac{2m}{I_T}\;\frac{r_1\,(r_1 + r_o)^2 + r(r + r_o)^2}{r + r_o} + \frac{r_1 - r}{r_1 + r_o}$$

Thus, the limiting frequencies are:

$$\lim_{\rho\to 0} \omega_1 = \omega_2 = \omega_3 = 0$$

$$\lim_{\rho\to 0} \omega_4 = \omega_o\sqrt{\left(\frac{2m}{M}\frac{r_o}{r_1}\right) / \left(1 - \frac{2m}{\mathcal{M}}\right)} \simeq \omega_o\sqrt{\frac{r_o}{r_1}}\left(1 + \frac{m}{\mathcal{M}}\frac{r + r_o}{r_o} + \ldots\right.$$

$$\lim_{\rho\to 0} \omega_5 = \omega_o\sqrt{\left(\frac{2m}{M} + \frac{r_o}{r}\right) / \left(1 - \frac{2m}{\mathcal{M}}\right)} = \omega_o\sqrt{\frac{r_o}{r}}\left(1 + \frac{m}{\mathcal{M}}\frac{r + r_o}{r_o} + \ldots\right)$$

$$\lim_{\substack{\rho\to 0\\ r_1\to r}} \omega_6 = \sqrt{\frac{p}{a}} = \omega_o\sqrt{\frac{r_o}{r}}$$

$$\lim_{\substack{\rho\to 0\\ r_1\to r}} \omega_7 = \omega_o\sqrt{\frac{r_o}{r}\,\frac{I_T}{I_o}}$$

CHAPTER 8

ENGINEERING ASPECTS

For computational analysis of the satellite dynamics, the usual inertia, damping, and stiffness coefficients of the interacting components must be known. The figures selected here have been obtained from the principal investigators and from Boeing Wire Boom Feasibility and Mass Properties reports. [1, 2]

8.1 General Physical Parameters

The satellite hub weight is four-hundred ninety (490) pounds. The moment of inertia about the spin axis is one-hundred fifteen (115) slug ft^2. For out-of-plane oscillations, transverse moments of inertia about the boom axes (i.e. about hub diagonals) are taken to be eighty (80) slug ft^2.

Each tip mass is a three (3) inch diameter sphere with a weight of two (2) pounds.

The wire connecting each tip mass to the center body is RAYCHEM TRIAX Cable EPD-1763, Kynar insulated. The wire diameter is 0.2 cm. and weight is 0.00635 lbs/ft.

The wire booms are effectively anchored at points which are 2.67 ft. from the hub center or spin-axis. This is the hub radius r_o in the analysis. Wire boom movements are assumed to occur through bending of the wire at these anchor points.

8.2 Stiffness and Damping Parameters

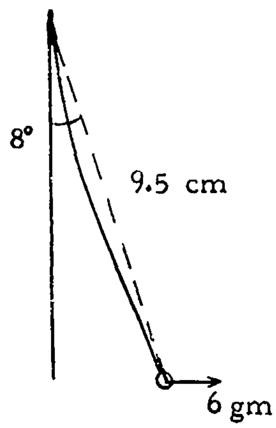
Four (4) physical parameters are required to define the coupling and interaction of the wire booms with the satellite hub. These are:

1) Wire stiffness or angular spring constant S
2) Internal wire damping due to inelastic bending of the wire K_ϕ
3) Atmospheric drag damping
4) A special Coulomb damper designed to increase wire boom energy dissipation through the hub

In order to determine these parameters, except for atmospheric drag, special experiments were conducted by the principal investigators and disseminated to the writers for the purpose of this study.

A summary of the essential results and a derivation of the coefficients is given below. Determination of these coefficients is rendered extremely difficult due to the presence of gravitational forces and large atmospheric drag, factors that will not be present in flight. These large effects are discounted by studying the result of appropriate changes in the experiment conditions.

8.2.1 Wire Stiffness S



A spring scale reading of 6 gm at a 9.5 cm distance from the suspension point gives a deflection of 8 degrees in the wire

$$S \times 8/57 = 6 \times 980 \times 9.5$$

$$\therefore S \approx 4 \times 10^5 \text{ gm. cm}^2/\text{sec}^2$$

8.2.2 Two types of pendulum damping experiments were conducted. One was with short wire lengths where air drag could be neglected. The other was with wire lengths comparable to those in the actual case. Here air damping is predominant, and the Coulomb friction damper was also experimented with. In all cases a tip mass of 220 gms. was used to approximate the force of a 2 lb tip mass under actual operating conditions (30 - 50 ft. wire length, 3 - 4 RPM centrifugal force field).

The usual second order differential equation gives the relationship between the desired parameters and the observations:

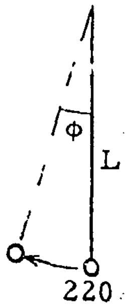
$$(m + \frac{\rho r}{3}) r^2 \ddot{\phi} + K_\phi \dot{\phi} + \ldots . \phi = 0$$

In the form $\ddot{\phi} + 2\beta\dot{\phi} + \ldots . \phi = 0$, we have the damping time constant $\mathcal{T}$ given by $\frac{1}{\mathcal{T}} = \beta = \frac{1}{2} K_\phi / (mr^2 + \rho r^3/3)$

If an experiment is conducted for T secs during which n complete oscillations occur, with initial amplitude ϕ_o and final amplitude ϕ_n we further have

$$\phi_n \approx \phi_o e^{-\beta T} \qquad \text{OR,} \quad \beta = \frac{1}{T} \ln \frac{\phi_o}{\phi_n}$$

a) Test 1: L = 50.135 cm



Time (sec)	ϕ^o	No. of Swings	β sec^{-1}	K_ϕ $gm.cm^2/sec$
0	10.69	0	-	-
42.7	6.08	30	.0131	14,250
52.75	5.32	37	.01314	14,300

Test 2: L = 26.535 cm

Time (sec)	ϕ^o	No. of Swings	β sec^{-1}	K_ϕ $gm.cm^2/sec$
0	16.13	0	-	-
30.0	6.38	30	.031	9,600
49.0	3.185	49	.0331	10,200

The higher damping figure in Test 1 may be attributed to the air damping with the longer wire length. However, this effect must be fairly small since the calculated K_ϕ does not decrease as the amplitudes diminish.

The wire damping due to angular bending about the suspension point is estimated at $K_\phi = 10,000$ gm cm^2/sec.

b) This experiment attempted to differentiate between the effects of the special Coulomb friction damper, the internal wire damping with the normal suspension, and the ever-present air damping, by running a special test with a knife edge suspension and no damper.

In all cases the pendulum period was 6 secs. corresponding to a pendulum length of 892 cm or 352.35",

$$m\,r^2 + \frac{\rho r^3}{3} = 1.82 \times 10^8 \text{ gm cm}^2$$

	Test 1 Normal suspension incl. damper and wire damping K_ϕ	Test 2 Normal Suspension, no damper, but wire damping K_ϕ	Test 3 Knife edge No damper No wire damping
Amplitude at swing #8	4' 4.25"	4'8"	4'4"
Amplitude at swing #31	1'8"	1' 10.75"	1' 9.5"
$\beta = \frac{1}{6 \times 23} \ln \frac{\phi_o}{\phi_n}$ sec^{-1}	0.00695	0.0059	0.0064
Damping gm cm^2/sec	$k_D + k_\phi + k_A = 2.52 \times 10^6$	$k_\phi + k_A = 2.15 \times 10^6$	$k_A = 2.32 \times 10^6$

where k_ϕ is the wire damping, k_D is the Coulomb damping, and k_A is the effective air damping.

Comparison of Test 2 with Test 3 results in a negative k_o which is impossible. It is concluded that k_ϕ is so much less than the air damping k_A that tests 2 and 3 are equivalent. The previous experiment gave $k_\phi = 1.0 \times 10^4$ gm cm^2/sec which confirms the conclusion.

Test 1 then gives k_D in the range 2.0×10^5 to 3.7×10^5 gm cm^2/sec. This result and therefore all of this second experiment is useful only as a reasonableness check. The Coulomb damper is an order of magnitude less than the sea level atmospheric damping. Further, while the initial swings were in excess of 8^o angular deflection, the amplitude at swing number 31 (about 3^o) shows that the damper is barely active. A more exact representation of the Coulomb damper is provided in part (4) below.

8.2.3 It is readily established that atmospheric drag should be a negligible factor in the dynamics of the 1975 satellite. A conservative calculation is carried out below to compare atmospheric drag torque on a wire boom with the wire damping torque.

3" diameter sphere at tip, booms of 50' length and 0.2 cm wire diameter. Assume 200 km satellite altitude, winter season, 1700° K exospheric temperature. Also assume that the booms are oscillating with an amplitude of 0.2 radians at a frequency of 0.03 cps.

Sphere cross-section = 45.5 cm^2
Wire cross-section = 305 cm^2

Maximum velocity at tip mass:

$$= 50' \times 12'' \times 2.54 \times 0.2 \times .03 \times 2\pi$$

$$= 57.5 \text{ cm/sec}$$

Atmospheric density $\rho = 4 \times 10^{-13} \text{gm/cm}^3$

$$\text{Drag force} = \frac{C_D \, \rho A}{2} \, V^2$$

For drag coefficient $C_D = 2$, and lumping the cross-sectional area, Maximum drag force

$$= \frac{2 \times 4 \times 10^{-13} \times 350}{2} \times (57.5)^2$$

$$= 4.6 \times 10^{-7} \text{ gm cm/sec}^2$$

Maximum drag torque

$$= 4.6 \times 10^{-7} \times 50 \times 12 \times 2.54 = 7 \times 10^{-4} \text{ gm cm}^2/\text{sec}^2$$

Comparing with wire damping torque:

Maximum angular velocity $\dot{\phi}$

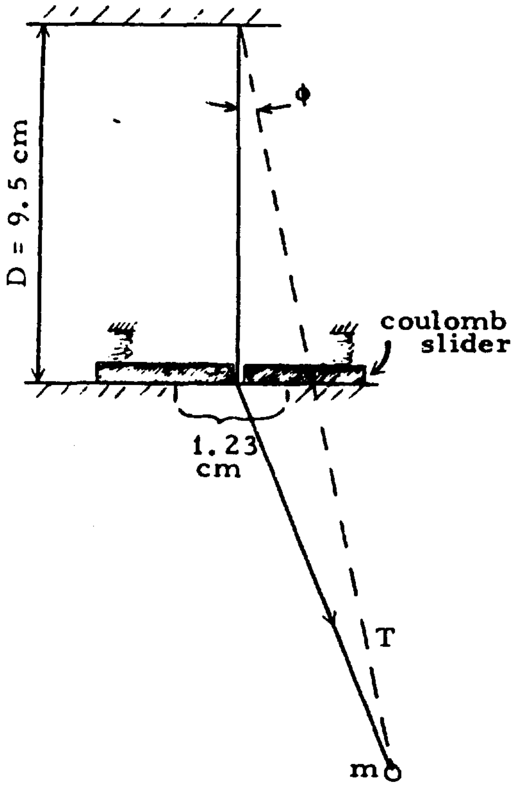
$$= 0.2 \times .03 \times 2\pi = 0.0375 \text{ rad/sec}$$

Maximum torque due to wire damping alone

$$= k_\phi \dot{\phi} = 10^4 \times 0.0375 = 375 \text{ gm cm}^2/\text{sec}^2$$

Therefore atmospheric drag torque is extremely small, even when compared with other low forces.

8.2.4 The Coulomb friction damper



The Coulomb slider has been tested and requires a force equal to 4 gms to overcome friction and maintain a sliding motion. The maximum end to end movement of the wire at the slider, and thus also the slider is 1.23 cm. The distance D between the point of suspension and the slider is 9.5 cm.

Since the wire boom length is in general so much greater than the distance D, the angle ϕ between the wire and the normal may be taken to be the same at the suspension or at the slider. For a tension T in the wire, the lateral force on the slider is given by T sin ϕ. As an example, a 2 lb tip mass on a 50' wire spinning at 3 RPM will overcome the coulomb friction when the wire deflection exceeds 1.7°.

$$\phi_c = \sin^{-1}\left(\frac{4 \times 980}{T}\right) = 1.7^{\circ}$$

If the coulomb slider comes to rest for some wire deflection ϕ_1, it would move again only when the angle $\phi > \phi_1 + \phi_c$ or $< \phi_1 - \phi_c$ provided of course that the limit stops permit. Note that when the slider reaches the limit stop for $\phi > 3.7 + \phi_c$ (i.e. $\tan^{-1}\left(\frac{.615}{9.5}\right) + \phi_c$), the damper becomes inactive. The non-linear characteristics described are implemented in the program.

A reasonableness check for the value of k_D obtained in part (2b) above with the characteristics given here can be made using energy dissipation considerations:

The pendulum experiment in 2 had a 6 sec. period and an average amplitude of 3 feet or 0.1 radian. Energy dissipated per cycle

$$= \int k_D \dot{\phi} \, d\phi$$

Let $\phi = .1 \sin t$. Then $d\phi = .1 \cos t \; dt$

$\therefore$ Energy dissipated per cycle

$$= \int 3 \times 10^5 \times .01 \cos^2 t \, dt \approx 10^4 \text{ gm cm}^2/\text{sec}^2$$

Energy loss in the Coulomb damper per cycle

$$= \text{force } \times \text{distance}$$

$$= 4 \text{ gm} \times 980 \text{ cm/sec}^2 \times 1.23 \text{ cm} \times 2$$

$$\approx 10^4 \text{ gm cm}^2/\text{sec}^2$$

This confirms that the characteristics determined above should provide a consistent set of parameters for the calculation of the S3-2 satellite wire boom dynamics.

8.3 Estimate of Performance -- Damping of Large Disturbances

For moderate deflections of the wire booms, the behavior of the satellite will depend upon the oscillation modes that are excited. Based on whether there is translation of the center body different frequencies will be evidenced. The damping is amplitude and frequency dependent; when the amplitude drops below ϕ_c i.e. the value required to cause sliding friction, the oscillations will diminish very slowly.

When the initial amplitude for each boom is large enough to ensure maximum damper action ($>5.4^o$ in the example in part 4 above), an estimate may be made for the rate of decay of the oscillations. Assume 50 foot boom deployments, 0.2 rad initial amplitudes, and an oscillation frequency of 0.03 cps.

Maximum angular velocity of each boom

$= 0.2 \times 0.03 \times 2\pi = 0.0375$ rad/sec

$\therefore$ Energy in each boom

$$= \frac{1}{2} (mr^2 + \rho r^3/3)\ \omega^2$$

$$= \frac{1}{2} (910 \times 1560^2 + 0.0945 \times 1560^3/3)\ 0.0375^2$$

$$= 1.65 \times 10^6 \text{ gm cm}^2/\text{sec}^2$$

Energy dissipation per cycle in the damper

$$= 10^4 \text{ gm cm}^2/\text{sec}^2$$

Therefore the damping rate is initially adequate to dissipate all the energy in 165 cycles or $165/0.03 = 5500$ secs.
Thus the damping time constant for large disturbances is approximately one earth orbit.

CHAPTER 9

DISCUSSION OF RESULTS

To recapitulate the results, the main features of wire boom-satellite dynamics are listed in this chapter. In view of their general nature, they should be useful for a wide class of satellite experiments featuring deployable/retractable wire booms.

9.1 Main Features of Analytical Results

1) Satellite hub spin slows down as booms are being deployed and speeds up as booms are being retracted.

2) Hub spin is steady when the booms are oscillating completely out-of-phase with each other (in an uncoupled mode).

3) Hub spin rate oscillates with a frequency identical to that of booms oscillating in phase with each other (in a coupled mode).

These first three points are due to conservation of angular momentum.

4) The system is less stable during boom retraction. This can be understood by looking at Equ. (2-5) or (2-6), in which a negative $\dot{r}$ reduces or even reverses the sign of the damping term β . Thus wide deflections of booms occur when

$$\dot{r} + k/2mr < 0 \qquad \text{(uncoupled case)}$$

or

$$\dot{r} + kI_T/2mrI_o < 0 \qquad \text{(coupled case)}$$

5) Damping is prominent when booms are short. The damping term β (in Equ. 2-5 or 2-6) is proportional to r^{-2}. This effect is not due to atmospheric drag or Coulomb damper, but due to the property of the wire used for the booms.

6) The term responsible for forced oscillation due to deployment/retraction is prominent when booms are short, since the forcing term F(t) in Equ. (2-5) or (2-6) is proportional to the ratio $\dot{r}/r$.

7) The amplitude of boom oscillation after deployment/retraction period τ depends on the amplitude and velocity of boom oscillation at the moment of stopping deployment/retraction, because ϕ (t > τ) is proportional to $\phi(\tau)$ and $[\beta\phi(\tau) + \dot{\phi}(\tau)]$ in Equ. (2-10).

8) The uncoupled mode frequency is the lowest frequency encountered in any satellite-boom configuration. Presence of translation implies the existence of an uncoupled mode, and gives somewhat higher frequencies than if translation is ignored. The coupled mode frequency is always higher than the uncoupled frequency because of the higher level energy interaction with the hub. The coupled mode with symmetrical boom lengths is characteristically devoid of translation. Finally, out-of-plane frequencies are consistently higher than in-plane frequencies because the tip masses operate in a normal rather than in a radial field.

9) The frequencies of modes in hub spin plane are generally lower than the spin frequency unless the boom lengths are shorter than the hub radius. Out-of-plane modes have frequencies always higher than the spin. See Figs. 11 and 12.

10) Beat phenomenon appear in modes involving translational oscillation of hub. For 1975 satellite, beat periods are usually about one to two thousand seconds, which accounts for the fine splitting of spectral lines by $\Delta\omega \sim .001$ c.p.s.

11) If all booms are simultaneously deployed or retracted, the coupled mode is excited because symmetry allows no way to distinguish one boom from another so that the booms move together in a coherent pattern.

12) If some but not all booms are deployed or retracted, generally both uncoupled and coupled modes emerge. Booms being deployed lag behind in phase relative to those undeployed or retracted. The uncoupled mode is usually favored because of its lower frequency and therefore lower energy level.

13) A remark about translation should perhaps be mentioned. If there is no external force, the center of mass of an isolated system initially at rest should not run away. The (X, Y) that we use are the hub center coordinates. Thus, if initially

($t = 0$) the center of mass and hub center coincide, then final translational displacement ($t \gg \tau$) must oscillation around the center of mass, i.e., the initial hub center ($X = 0$, $Y = 0$). If initially ($t = 0$) the center of mass does not coincide with the hub center (as in some cases where booms are initially deflected), then although the translational displacement (at $t \gg \tau$) still oscillates around the center of mass, it does not oscillate about the initial hub center ($X = 0$, $Y = 0$).

14) Although oscillations out of the spin plane are expected to be insignificant due to the presence of an effective wobble damper (see Ref. 1), it is nevertheless interesting to estimate the effect of possible out-of-plane boom deflections ψ_i on inplane mode frequencies, and vice versa. A nonzero value of ψ effectively shortens the inplane boom length ($r_i \rightarrow r_i \cos \psi_i$) so that the total moment of inertia changes somewhat, and the potential energies are also affected. Since the potential energies come essentially from the cosine terms in the Lagrangian (see, e.g. the second page of Chapter 5 and the third page of Chapter 6), a deflection ψ out-of-spin-plane contributes to the inplane potential energy, for,

$$r_i \cos \phi_i \cos \psi_i = r_i \left(1 - (\psi_i^2 + \phi_i^2)/2 + \ldots \right)$$

for small angular deflections. Thus, inplane mode potential energy terms are located not only in the inplane vector space represented by V_{in} but also in the out-of-plane vector space V_{out}. The opposite is also true for out-of-plane modes. Therefore, a rigorous formulation of the inplane $\leftrightarrow$ out-of-plane modes should involve 14x14 matrices T and V. However, since ψ_i is expected to be small, the results obtained by considering disjoint inplane and out-of-plane vector spaces are certainly of satisfactory accuracy.

9.2 Remarks on Fourier Spectral Analysis

Figure 9 gives the Fourier transforms of several time series of fundamental importance.

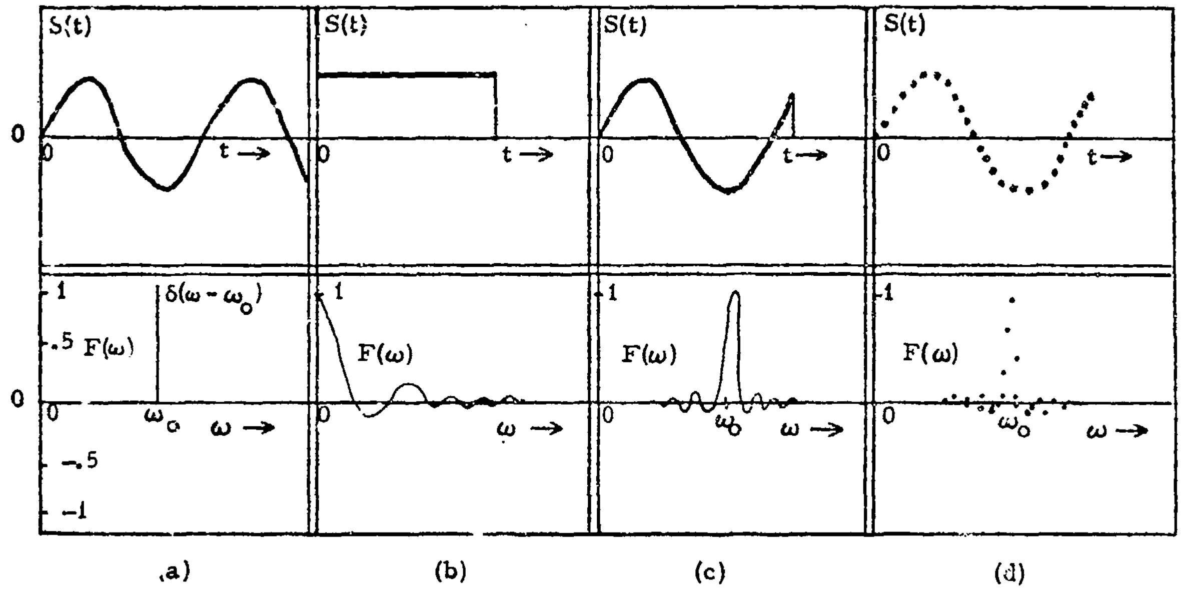


Figure 9. Fourier Transform F(t) of Four Time Series S(t) of Fundamental Importance

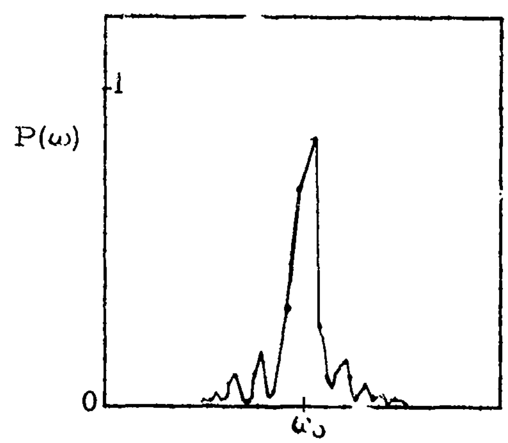


Figure 10. Power Spectrum of Case (d) in Fig. 9

In Fig. 9(a), the Fourier transform of an infinite train of sine wave $\sin(\omega_0 t)$ is just a $\delta(\omega - \omega_0)$ function. The fourier transform of a pulse step function is a $(\sin \omega_0 T) / (\omega_0 T)$ diffraction function [Fig. 9(b)]. Thus, a finite sine wave train of length T [Fig. 9(c)] gives a diffraction pattern around the ω_0 (power) spectral line with sidelobe maxima located at

$$\omega_0 \pm (n + 1/2) / T \qquad (n = 1, 2, 3, \ldots)$$

and sidelobe minima located at

$$\omega_0 \pm n/T \qquad (n = 1, 2, 3, \ldots.)$$

The time series length T is also called window width. The wider a window is, the less diffraction appears. If two spectral lines are too near each other, interference of diffraction patterns may occur.

In this report, most of the time series for spectral analysis have a window width of 2048 sec. A sine wave $\sin \omega_0 t$ of such length would give spectral sidelobe maxima at $\omega_0 \pm 0.000488\ (n + 1/2)$ c. p. s. and minima at $\omega_0 \pm 0.000488\ n$ c. p. s. $(n = 1, 2, \ldots)$. Also, in this report, most of deployment/retraction periods considered are 200 sec. A pulse step function of 200 sec. period would give spectral sidelobe mazima at $0 \pm 0.005\ (n + 1/2)$ c. p. s., minima at $0.005\ n$ c. p. s. $(n = \pm 1, \pm 2, \ldots)$, and a main peak at 0 c. p. s. However, if the pulse is not perfectly constant due to boom length change, a splitting of spectral line occurs, shifting the main peak slightly away from 0 c. p. s.

In Fig. 10, a situation of peak truncation is shown, where the actual peak happens to be in between two output points, due to finite resolution:

$\Delta \omega = \pm 1 / 2N \Delta T$, where N is the number of points to be transformed, and ΔT is the sampling rate.

The fast Fourier transform algorithm as developed by Cooley and Tukey [10] uses 2^N input points. For example, a 2048 sec. time series of sampling rate $\Delta T = 1$ sec. would give 2^{11} input points for fast Fourier transform, with a resolution: $\Delta \omega = \pm 0.00025$ c. p. s.

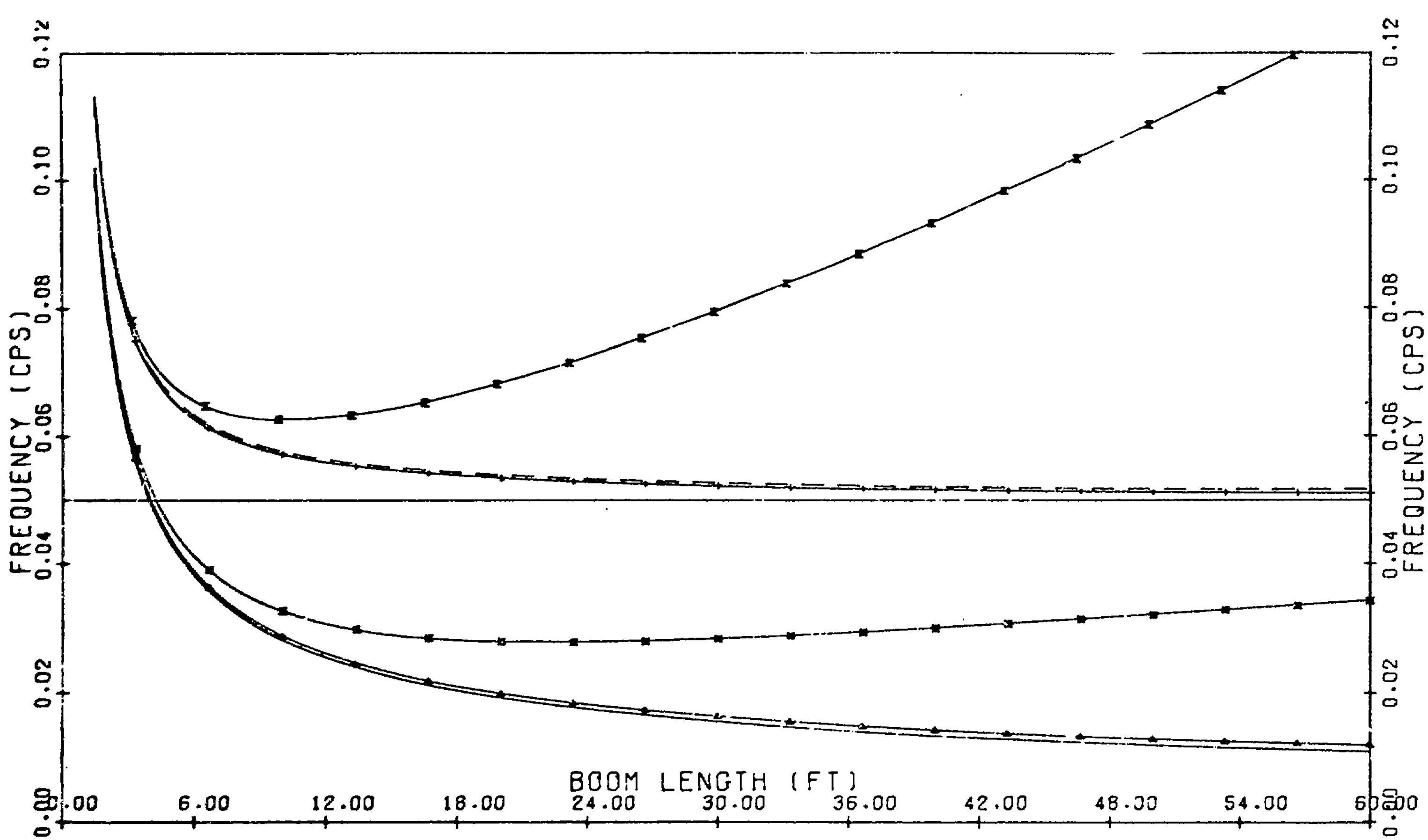


Figure 11. Non-trivial Harmonic Frequencies of Satellite System spinning at 3 RPM. The curves, in the order of descending frequency, represent out-of-plane coupled mode, out-of-plane jelly-fish mode, out-of-plane uncoupled saddle mode, in-plane coupled mode, inplane uncoupled mode with translation, and inplane uncoupled mode, respectively. The horizontal line at 0.05 c.p.s. is the hub spin frequency.

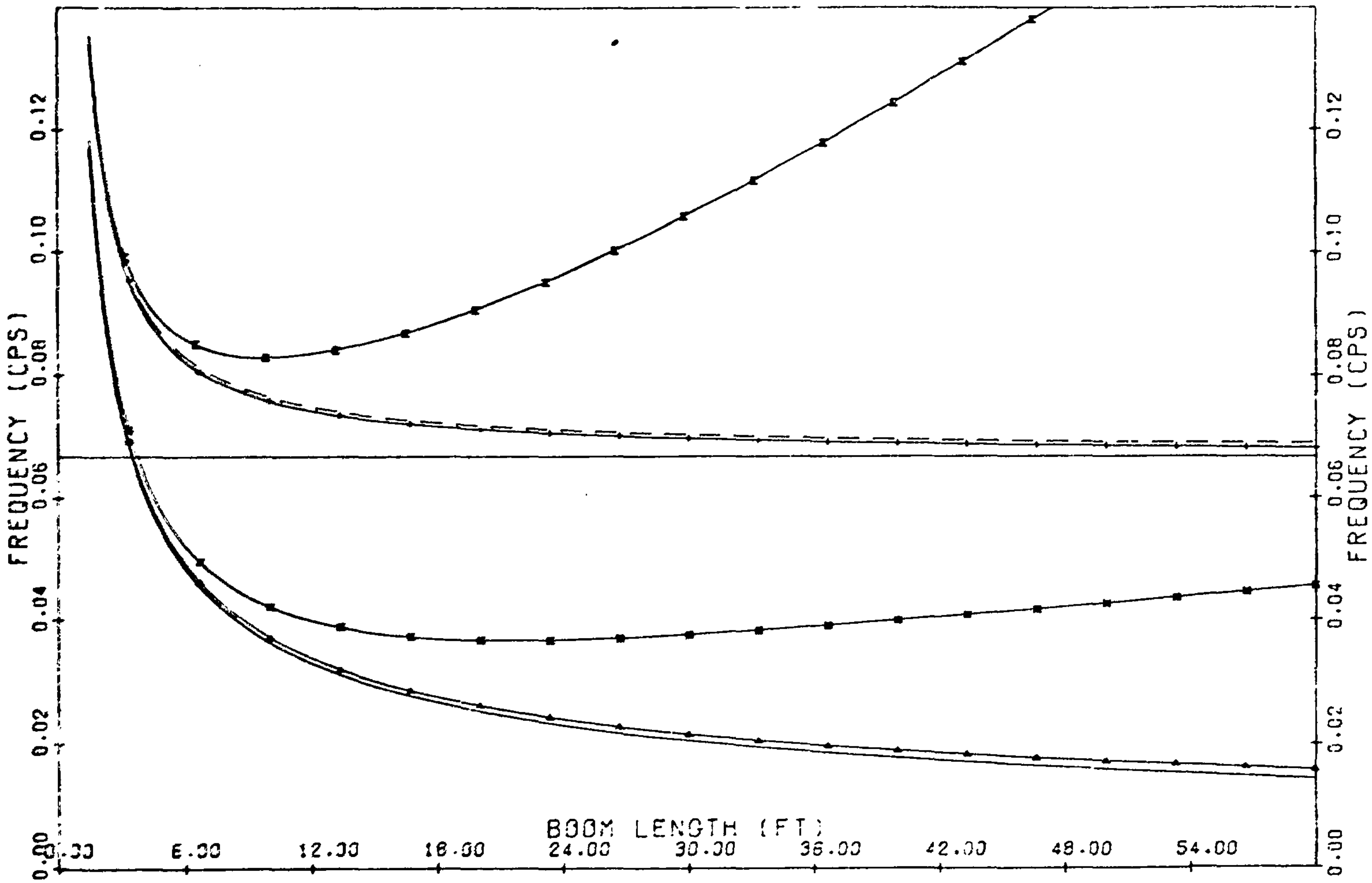


Figure 12. Nontrivial Harmonic Frequencies of Satellite System spinning at 4 RPM. The curves, in the order of descending frequency, represent out-of-plane coupled mode, out-of-plane jelly-fish mode, out-of-plane uncoupled saddle mode, inplane coupled mode, inplane uncouped mode with translation, and inplane uncoupled mode, respectively. The horizontal line at 0.0667 c.p.s. is the hub spin frequency.

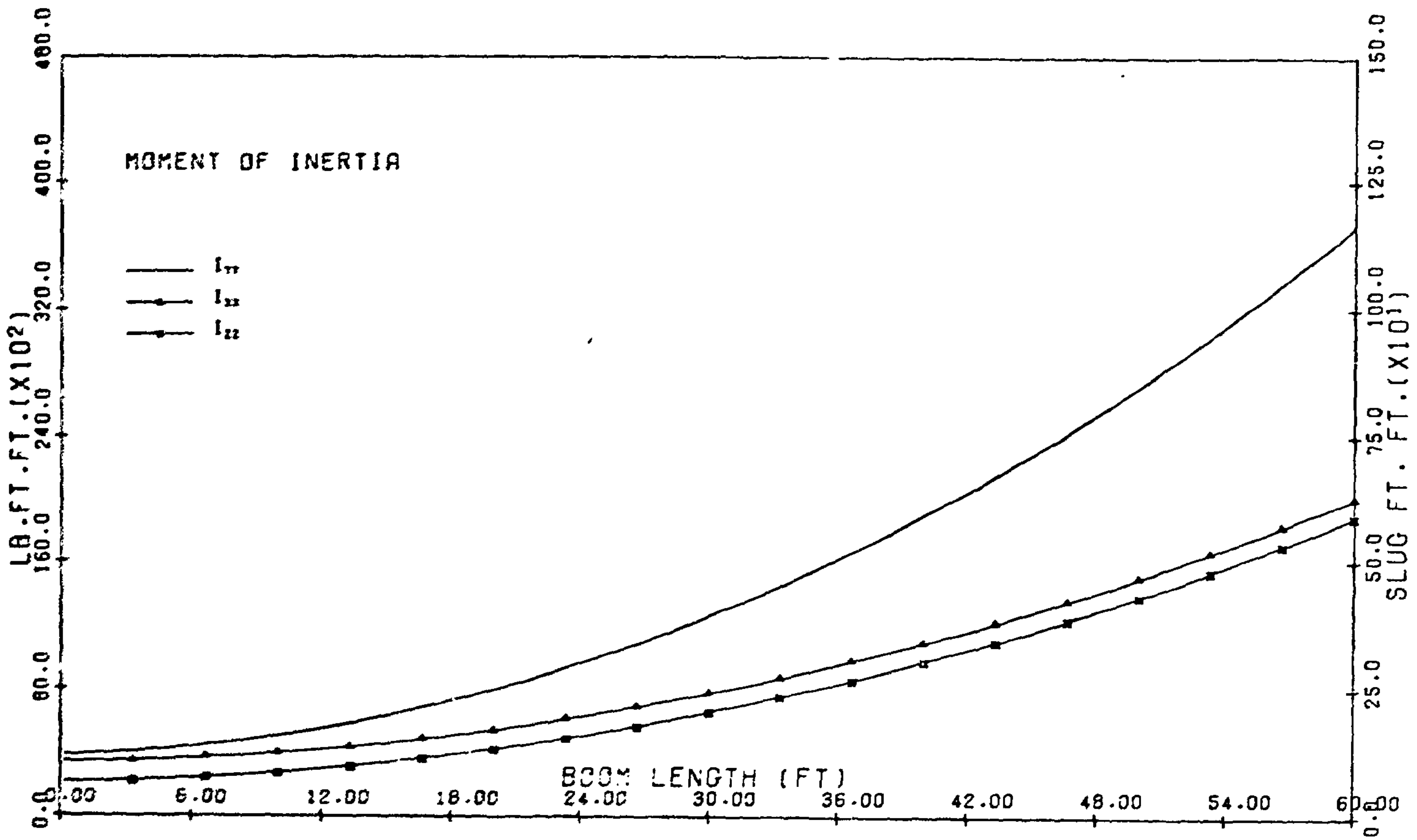


Figure 13. Moments of Inertia about the three principal axes. The one about the spin axis goes up faster as a function of boom length, because all four booms contribute in contrast to only two booms contributing to I_{xx} and I_{zz}.

Table 1. Normal Modes of Satellite with Four Equal Length Wire Booms - Neglegible Translational Oscillation of Hub

Classification of Mode	Shape of Mode (Characterized by the relative amplitudes of harmonic variables)							Harmonic Frequency	Limiting value of Harmonic Frequency: zero wire mass and spring constant
	ϕ_1	ϕ_2	ϕ_3	ϕ_4	σ	X'	Y'		
Uncoupled Mode	0	1	0	-1	0	0	0	$(p/a)^{1/2}$	$\omega_o (r_o/r)^{1/2}$
Uncoupled Mode	-1	0	1	0	0	0	0	$(p/a)^{1/2}$	$\omega_o (r_o/r)^{1/2}$
Uncoupled Mode	-1	1	-1	1	0	0	0	$(p/a)^{1/2}$	$\omega_o (r_o/r)^{1/2}$
Coupled Mode	1	1	1	1	$-\frac{4b}{I}$	0	0	$(p/a)^{1/2} \Big/ (1 - 4b^2/aI)^{1/2}$	$\omega_o (r_o I/r I_o)^{1/2}$
Pure Rotation Mode (Trivial)	0	0	0	0	1	0	0	0	0

Table 2. Normal Modes of Satellite with Four Equal Length Wire Booms Including Translational Oscillation of Hub

Classification of Mode	Shape of Mode (Characterized by the relative amplitudes of harmonic variables) ϕ_1	ϕ_2	ϕ_3	ϕ_4	θ'	X'	Y'	Harmonic Frequency	Limiting value of Harmonic Frequency: zero wire mass and spring constant
Pure Uncoupled Mode	1	-1	1	-1	0	0	0	$(p/a)^{1/2}$	$\omega_o(r_o/r)^{1/2}$
Uncoupled Mode with Translation	1	1	-1	-1	0	F	-G	$[(p/a)(1+\frac{2q}{p})/(1-\frac{2(mr+\rho r^2/2)^2}{\pi(mr^2+\rho r^3/3)})]^{1/2}$	$\omega_o(\frac{r_o}{r})^{1/2}[1+\frac{m}{M}\frac{r+r_o}{r_o}+\ldots]$ for $m(r+r_o) \ll Mr_o$
Uncoupled Mode with Translation	1	-1	-1	1	0	-G	-F	$[(p/a)(1+\frac{2q}{p})/(1-\frac{2(mr+\rho r^2/2)^2}{\pi(mr^2+\rho r^3/3)})]^{1/2}$	$\omega_o(\frac{r_o}{r})^{1/2}[1+\frac{m}{M}\frac{r+r_o}{r_o}+\ldots]$ for $m(r+r_o) \ll Mr_o$
Coupled Mode	1	1	1	1	-4b/I	0	0	$(p/a)^{1/2}/(1-4b^2/aI)^{1/2}$	$\omega_o(r_o I/r I_o)^{1/2}$
Pure Rotation Mode (Trivial)	0	0	0	0	1	0	0	0	0
Pure Translation Mode (Trivial)	0	0	0	0	0	1	0	0	0
Pure Translation Mode (Trivial)	0	0	0	0	0	0	1	0	0

Table 3. Normal Modes of Satellite with Pairwise Equal Length Wire Booms - Including Translational Oscillation of Hub

Classification of Mode	Shape of Mode (Characterized by the relative amplitudes of harmonic variables)							Harmonic Frequency	Limiting value of Harmonic Frequency: zero wire mass and spring constant
	ϕ_1	ϕ_2	ϕ_3	ϕ_4	θ'	X'	Y'		
Partially Coupled Mode	ν_-	1	ν_-	1	ν_+	0	0	$(p_1/a_1)^{1/2} \Big/ (1 + b_1 \nu_+ / a_1)^{1/2}$	$\omega_o (r_o/r)^{1/2}$ for $r = r_1$
Uncoupled Mode with Translation	0	1	0	-1	0	ξ_1	η_1	$\left[(p_1/a_1)(1+\frac{2q_1}{p_1}) \Big/ (1 - \frac{2(mr_1 + \rho r_1^2/2)^2}{\mathcal{M}(mr_1^2 + \rho r_1^3/3)}) \right]^{1/2}$	$\omega_o (r_o/r_1)^{1/2} [1 + \frac{m}{M} \frac{r + r_o}{r_o} + \ldots]$ for $m(r_1 + r_o) \ll M r_o$
Uncoupled Mode with Translation	-1	0	1	0	0	$-\eta$	ξ	$\left[(p/a)(1+\frac{2q}{p}) \Big/ (1 - \frac{2(mr + \rho r^2/2)^2}{\mathcal{M}(mr^2 + \rho r^3/3)}) \right]^{1/2}$	$\omega_o (r_o/r)^{1/2} [1 + \frac{m}{M} \frac{r + r_o}{r_o} + \ldots]$ for $m(r + r_o) \ll M r_o$
Coupled Mode	ν_+	1	ν_+	1	ν_-	0	0	$(p/a)^{1/2} \Big/ (1 + b \nu_-/a)^{1/2}$	$\omega_o (r_o I / r I_o)^{1/2}$ for $r = r_1$
Pure Rotation Mode (Trivial)	0	0	0	0	1	0	0	0	0
Pure Translation Mode (Trivial)	0	0	0	0	0	1	0	0	0
Pure Translation Mode (Trivial)	0	0	0	0	0	0	1	0	0

Table 4. Normal Modes Out-of-spin-plane of Satellite with Four Equal Length Wire Booms - Including Translational Oscillation of Hub

Classification of Mode	Shape of Mode (Characterized by the relative amplitudes of harmonic variables)							Harmonic Frequency	Limiting value of Harmonic Frequency: zero wire mass and spring constant
	ψ_1	ψ_2	ψ_3	ψ_4	θ_1	θ_2	Z		
Uncoupled Saddle Mode	-1	1	1	-1	0	0	0	$(q/a)^{1/2}$	$\omega_o[(r+r_o)/r]^{1/2}$
Coupled Mode	1	0	1	0	$-\frac{2b}{I_1}$	0	0	$(q/a)^{1/2}/(1-2b^2/aI_1)^{1/2}$	$\omega_o[(r+r_o)I_1/rI_o]^{1/2}$
Coupled Mode	0	1	0	1	0	$-\frac{2b}{I_2}$	0	$(q/a)^{1/2}/(1-2b^2/aI_2)^{1/2}$	$\omega_o[(r+r_o)I_2/rI_o]^{1/2}$
Jelly-Fish Mode	-1	-1	1	1	0	0	$4d/\eta$	$(q/a)^{1/2}/(1-4d^2/a\eta)^{1/2}$	$\omega_o[(r+r_o)(M+4m)/rM]^{1/2}$
Pure Rotation Mode (Trivial)	0	0	0	0	1	0	0	0	0
Pure Rotation Mode (Trivial)	0	0	0	0	0	1	0	0	0
Pure Translation Mode (Trivial)	0	0	0	0	0	0	1	0	0

Table 5. Frequencies generated by the fast Fourier transform of computer simulated satellite-wire boom dynamics compared to those calculated by using eigen-value analysis in harmonic approximation. Set #4 of the computer generated frequency spectrum shows some neighboring (and interfering) weak peaks.

Simulation Set	Dynamics				Simulation			Analysis			
	No.	Initial Length (m)	Final Length (m)	Initial Deflection (radians)	Frequencies Generated (cps)			Harmonic Frequencies (cps)			
#1	1	50	50	0.14	.0120	.0327		.0119 (uncoupled)	.0322 (coupled)		
	2	50	50	0.14	.0120	.0327					
	3	50	50	0.14	.0120	.0327					
	4	50	50	0.14	.0120	.0327					
#2	1	50	50	0.14	.0125	.0134	.0325	.0119 (uncoupled)	.0129 (uncoupled with translation)	.0322 (coupled)	
	2	50	50	0.07	.0125	.0132	.0325				
	3	50	50	0.	.0125	.0134	.0325				
	4	50	50	0.07	.0125	.0132	.0325				
#3	1	45	50	0.	.0120	.0322		.0119 (uncoupled)	.0322 (coupled)		
	2	45	50	0.	.0120	.0322					
	3	45	50	0.	.0120	.0322					
	4	45	50	0.	.0120	.0322					
#4	1	45	35	0.	.0134	.0148	.0306	.0139 (partially coupled)	.0143 (uncoupled)	.0151 (uncoupled with translation)	.0307 (coupled)
	2	45	45	0.	.0134	.0148	.0306				
	3	45	45	0.	.0126	.0148	.0306				
	4	45	45	0.	.0134	.0146	.0306				
#5	1	40	40	0.	.0212	.0317		.0211 (partially coupled)	.0319 (coupled)		
	2	40	35	0.	.0212	.0317					
	3	20	10	0.	.0212	.0317					
	4	20	30	0.	.0212	.0317					

9.3 Computer Simulation Results

The results of a number of digital computer simulation runs for satellite system 1975 with various given initial conditions are presented graphically. These results are the time series of the satellite system variables evaluated by solving a set of coupled differential equations [Equ. (3-8) to (3-11)], [Computer Program SATEDYN].

In the first set of simulation graphics [Fig. 14 (a-d)], two modes are shown: the coupled mode and the uncoupled mode without translation. They are excited by initial boom deflections: (0.14, 0, 0.14, 0) radians. No deployment/retraction of booms is involved, and the boom lengths are all equal (50 ft.). The presence of the coupled mode is indicated by the oscillating behavior of hub angular velocity (spin rate, i.e., $\omega(t) = \omega_0 + \dot{\theta}'(t)$) in Fig. 14 (a). The uncoupled mode prevails eventually because of its lower frequency (or energy). Since symmetry allows no way to distinguish boom 1 from boom 3, and boom 2 from boom 4, only the deflection of boom 1 and boom 2 (curve with asterisks) are shown [Fig. 14 (b)]. The predominant ucoupled mode gives a higher peak in the power spectrum plots of boom 3 [Fig. 14(c)] and boom 4 [Fig. 14 (d)]. The next higher harmonics are too weak to show up in the graphics but can be revealed by means of digital printouts (not shown).

Simulation graphics set #2 [see Fig. 15 (a-i)] shows the coupled mode and uncoupled modes with tranlational oscillations of the hub. The modes are excited by initial boom deflections (0.14, 0.07, 0, 0.07) radians. Again, no deployment/retraction of booms is involved, and the boom lengths are all equal (50 ft.). Oscillatory behavior of hub angular velocity [Fig. 15(a)] persists throughout the simulation period. Translational oscillations of the heavy hub are shown to be small in amplitude ($X = X_0 + X' \sim 1$ cm.) , Fig. 15 (b,)]. The initial hub center coordinates do not coincide with the center of mass because of the initial unsymmetrical boom deflections, and therefore the translational oscillations (X, Y) eventually do not center around the initial coordiates X =0, Y=0) of the center of the hub. Beat phenomenon can be seen in the oscillations of the booms 1 and 2 (curve with asterisks) [Fig. 15(d)] and those of the booms 3 and 4 (curve with asterisks) [Fig. 15 (e)]. The beats are responsible for the fine splitting of the uncoupled mode frequency [Fig. 15 (f-i)]. The inverse of the beat period is equal to the difference of splitted frequencies. The degeneracy of the space of uncoupled mode eigenfunctions has been removed by the slightly broken symmetry caused by the appearance of small amplitude hub translations.

Figure 14 (a-d)

Simulation graphics: Set #1
Satellite boom lengths: 50 ft. (all)
Deployment/Retraction: Nil
Initial Boom Deflections: (0.14, 0, 0.14, 0)

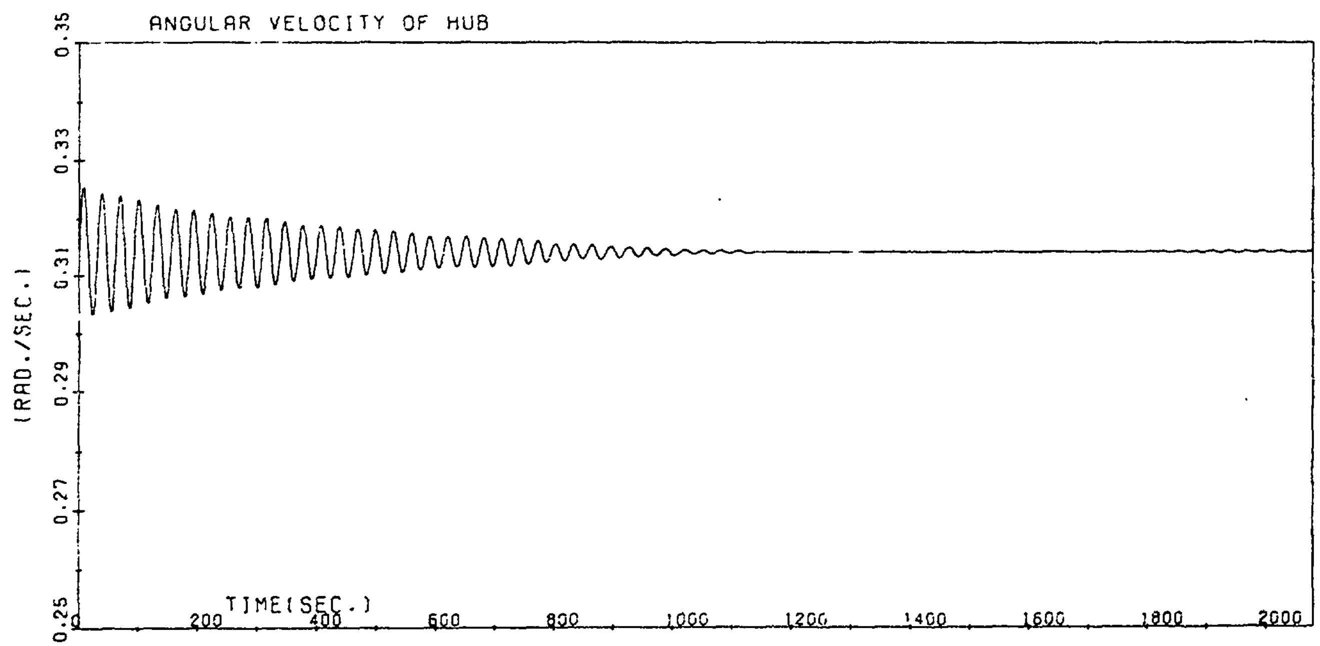


Figure 14(a). Angular velocity $[\omega(t) = \omega_b + \dot{\theta}'(t)]$ of hub as a time series. The presence of the coupled mode is indicated by the oscillatory behavior of the angular velocity.

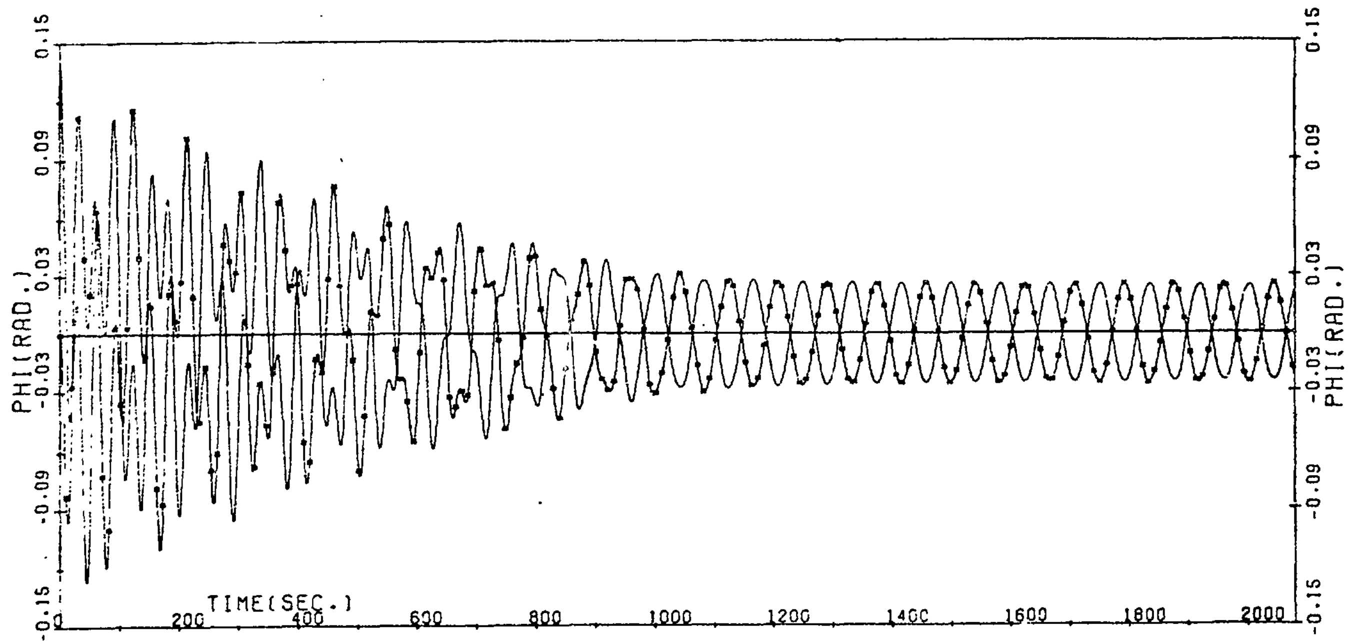


Figure 14(b). Deflections of boom 1 (or 3) and boom 2 (or 4) with asterisks labelling the latter curve. No translation of the hub is involved because of the symmetrical movements of the booms.

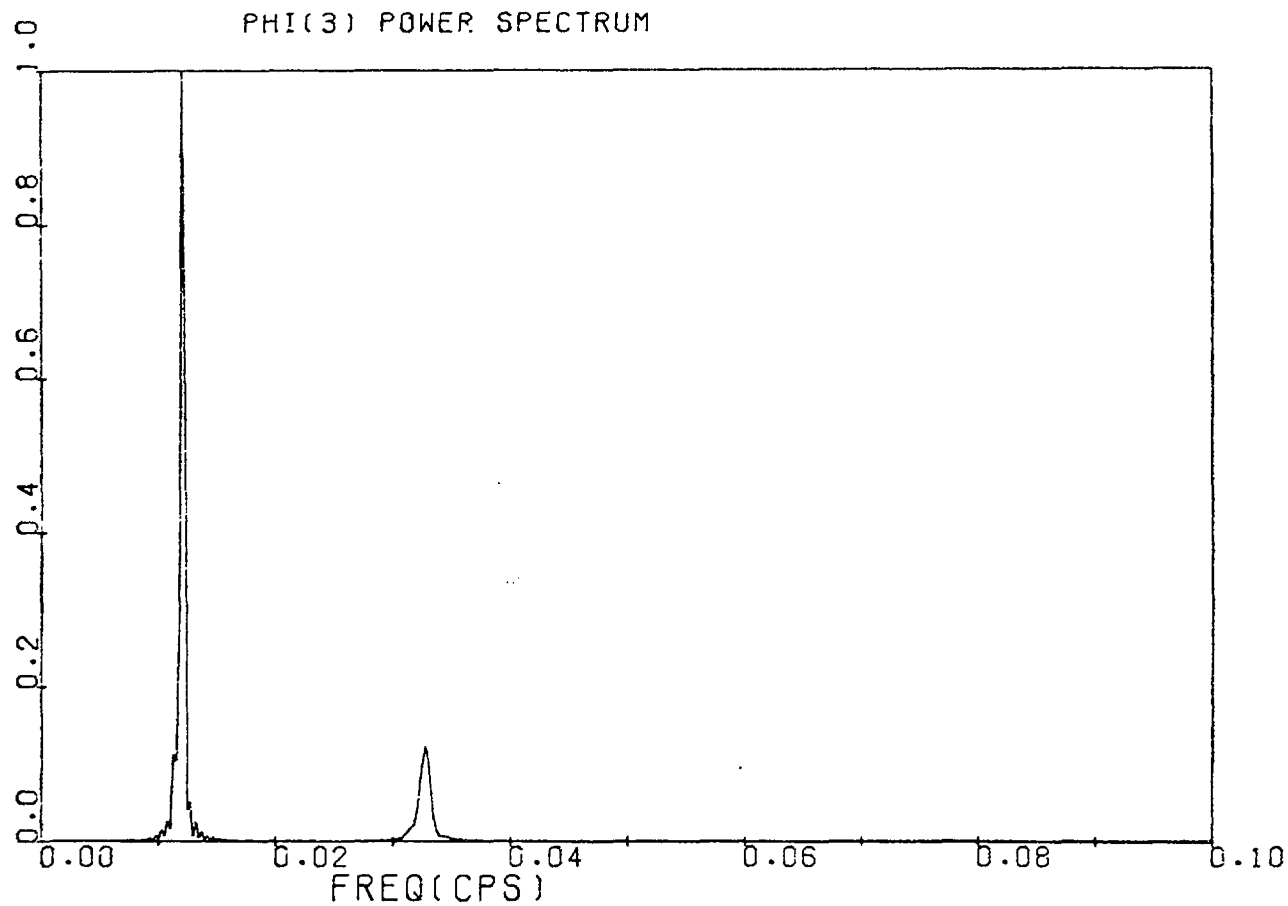


Figure 14(c). Power spectrum of boom 1 (or 3) oscillations. The lower frequency spectral line is the uncoupled mode frequency and the other is the coupled mode frequency. Higher harmonics are too weak to show up graphically.

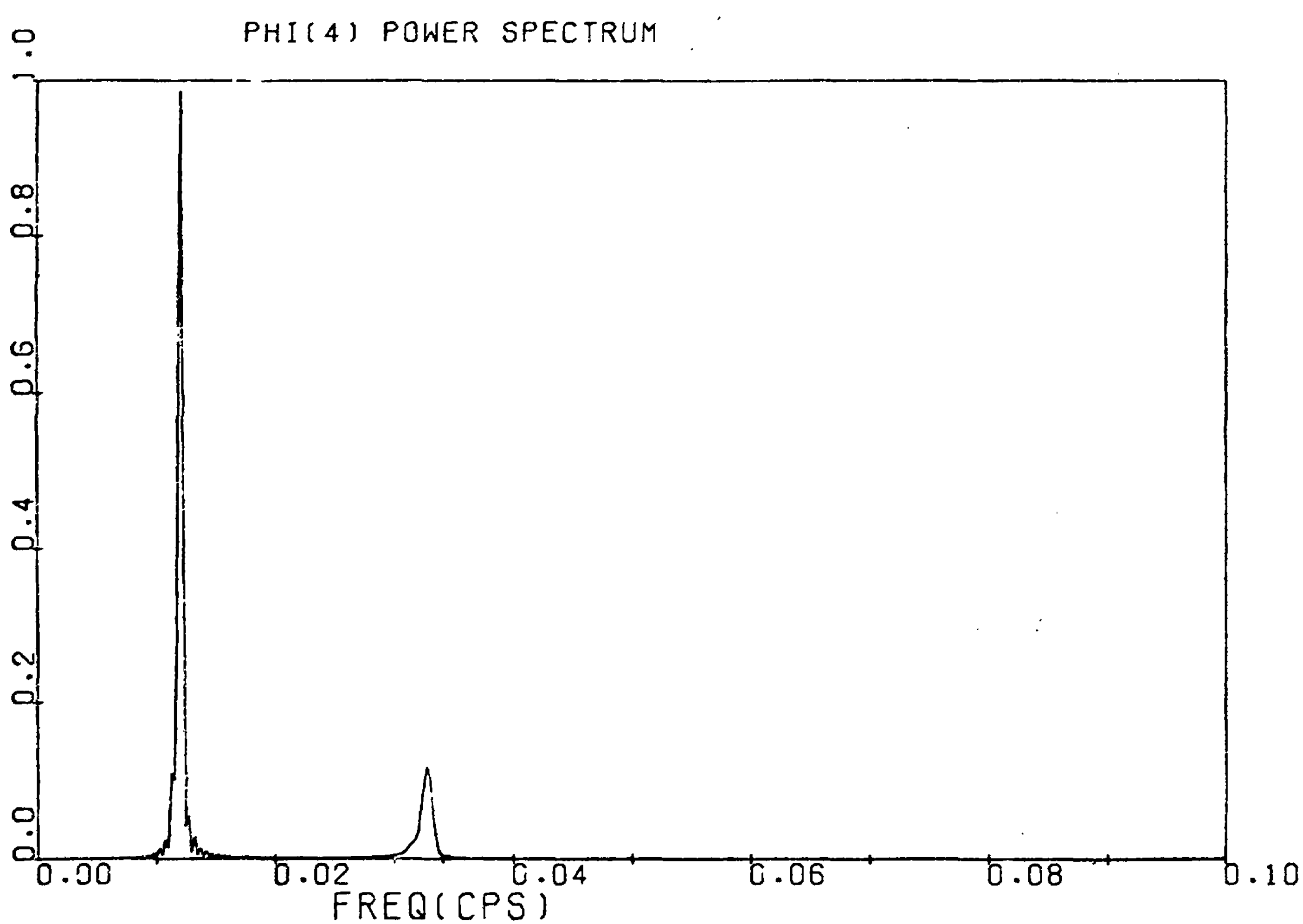


Figure 14(d). Power Spectrum of boom 2 (or 4) oscillations. See caption of the preceding figure.

Figure 15 (a-i)

Simulation graphics: Set #2
Satellite boom lengths: 50 ft. (all)
Deployment/Retraction: Nil
Initial Boom Deflections: (0.14, 0.07, 0, 0.07)

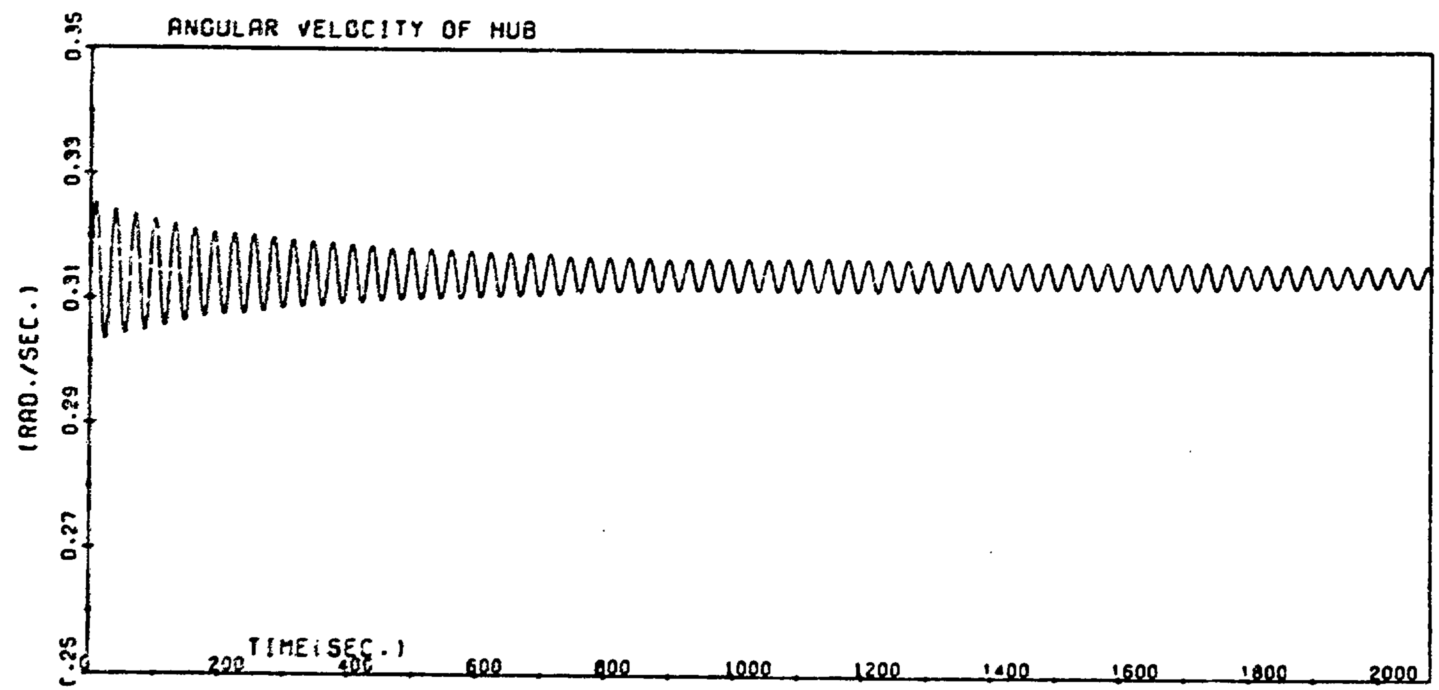


Figure 15(a). Angular velocity of hub, showing presence of the coupled mode.

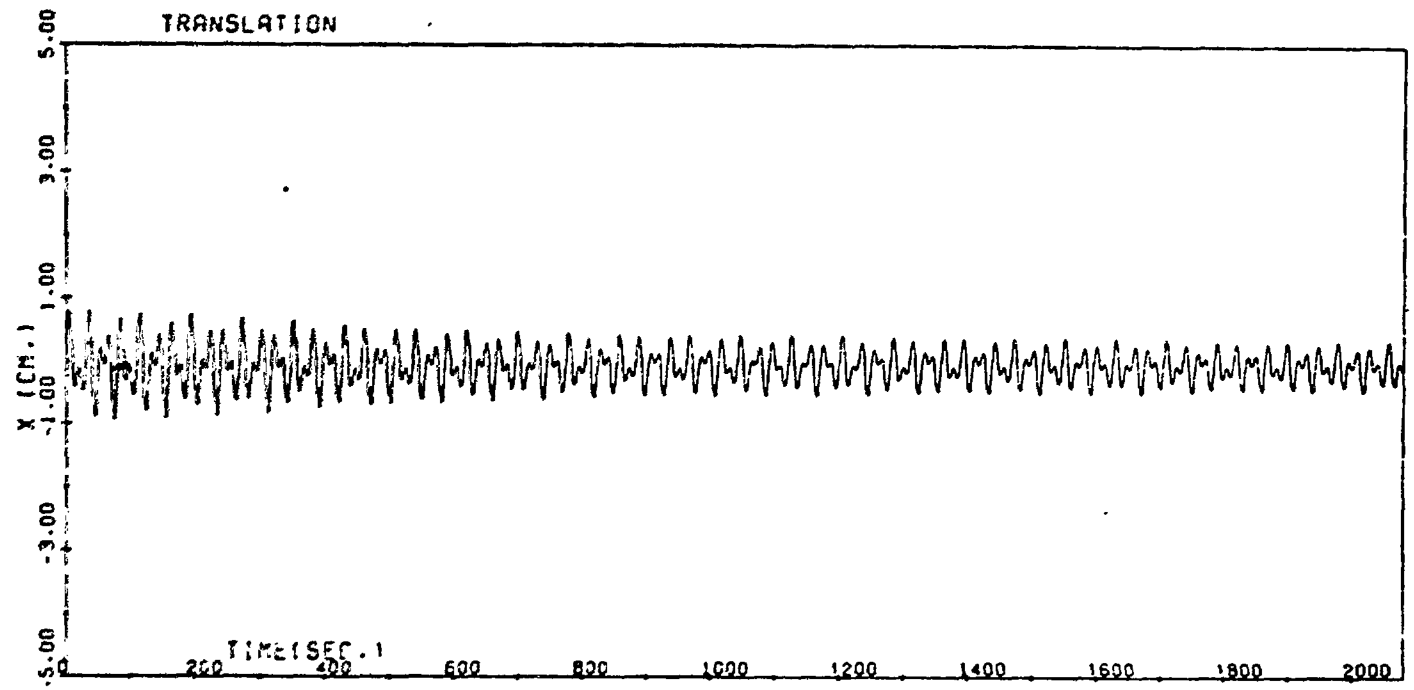


Figure 15(b). Translational Oscillations $[X(t)=X_o(t)+X'(t)]$ of the heavy hub. The origin (X=0, Y=0) is the position of hub center at t=0.

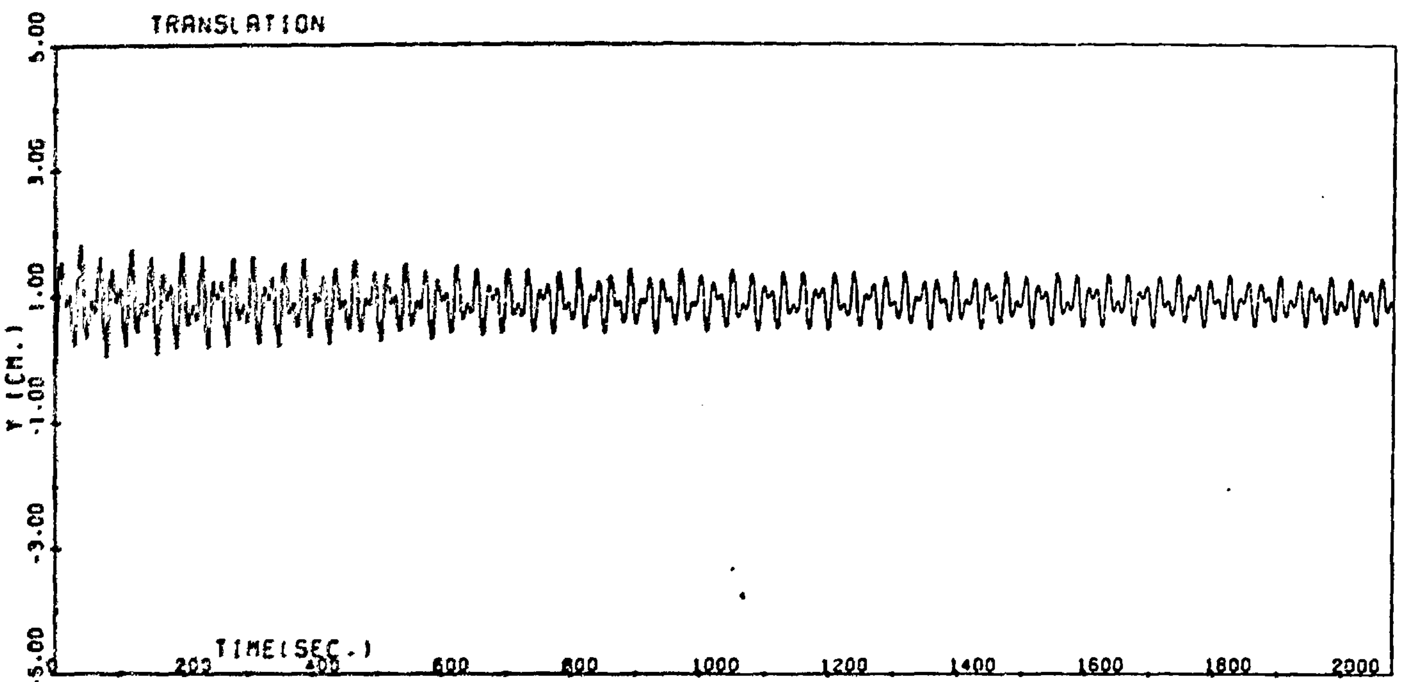


Figure 15(c). Translation oscillations [$Y(t) = Y_o(t) + Y'(t)$] of the heavy hub.

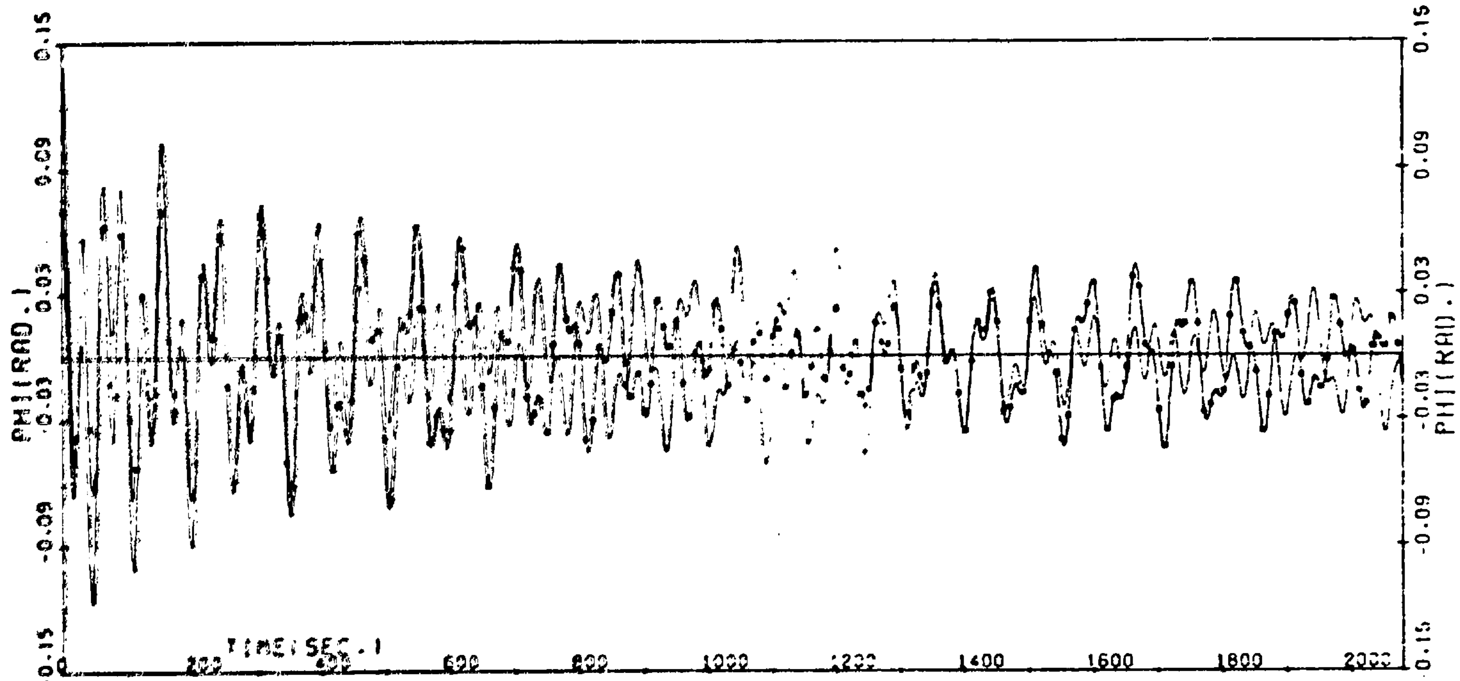


Figure 15(d). Deflections of boom 1 and boom 2 (with asterisks). The beats can be clearly seen.

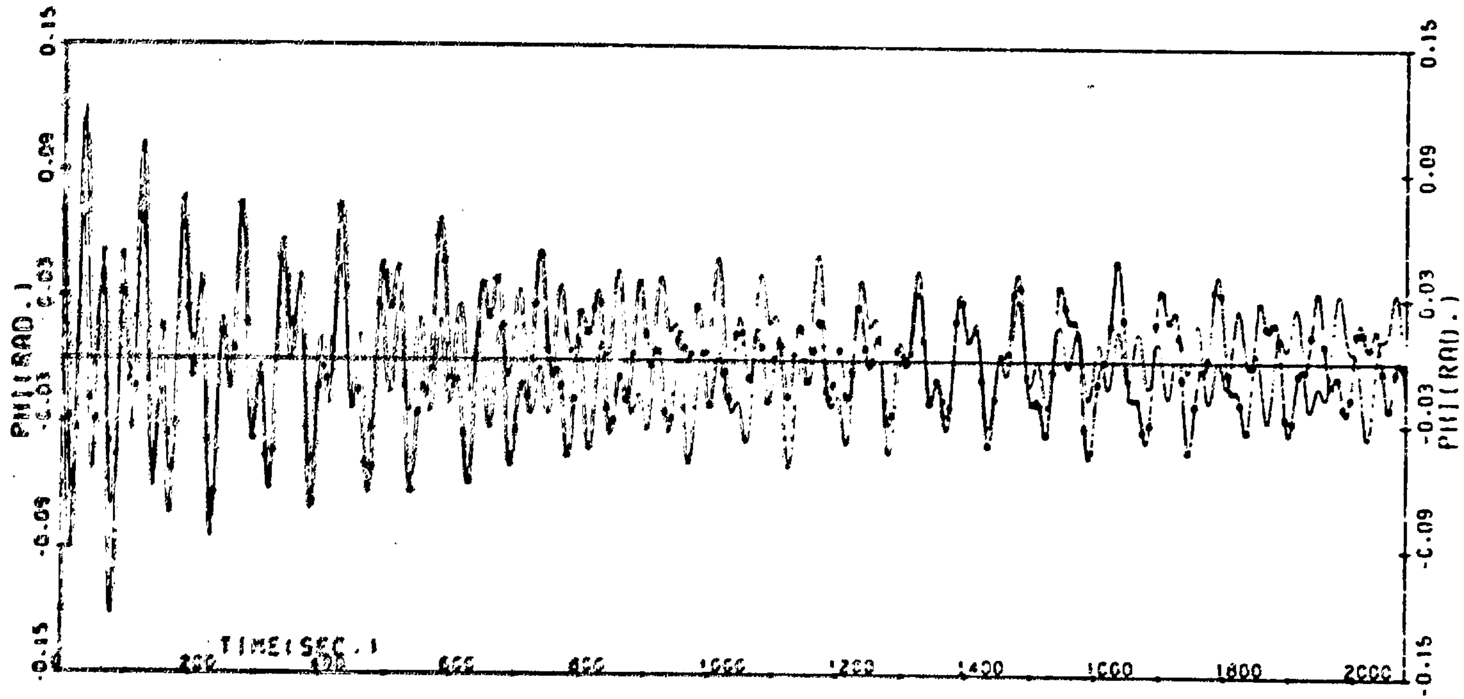


Figure 15(c). Deflections of boom 3 and boom 4 (with asterisks). The beats can be clearly seen.

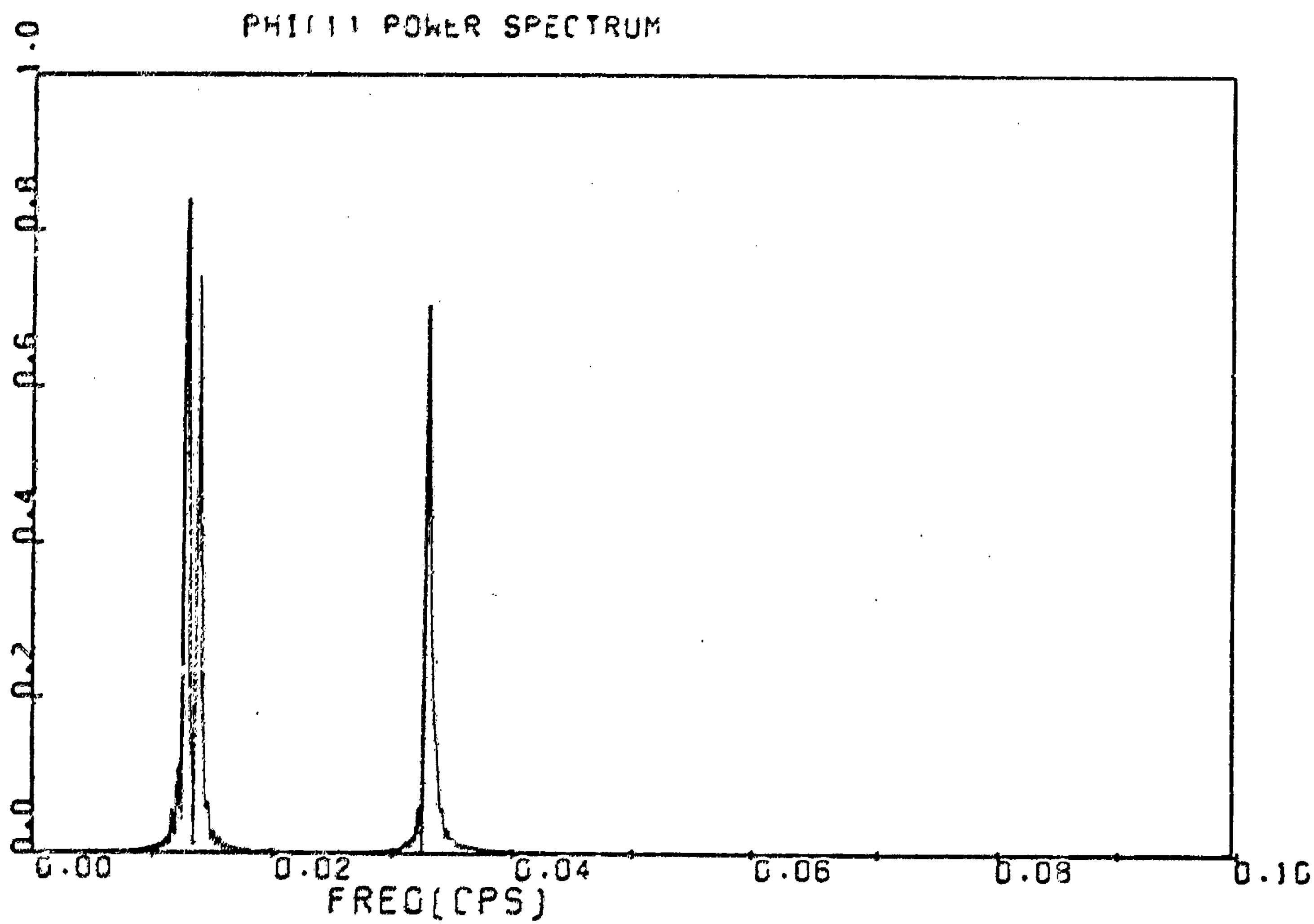


Figure 15(f). Power spectrum of boom 1 oscillations. Fine spectral splitting is due to beats, as a result of translational oscillations of hub.

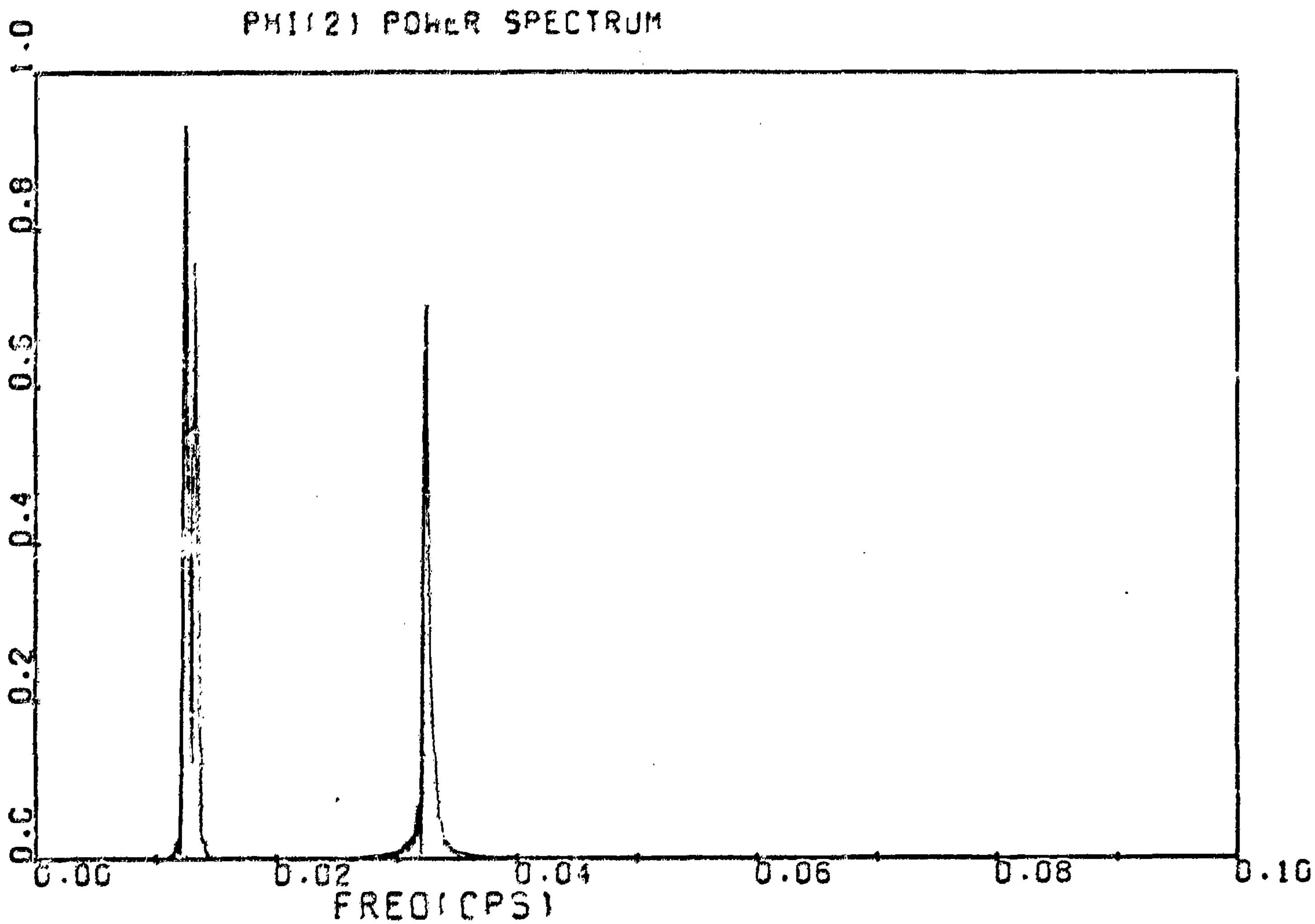


Figure 15(g). Power spectrum of boom 2 oscillations.

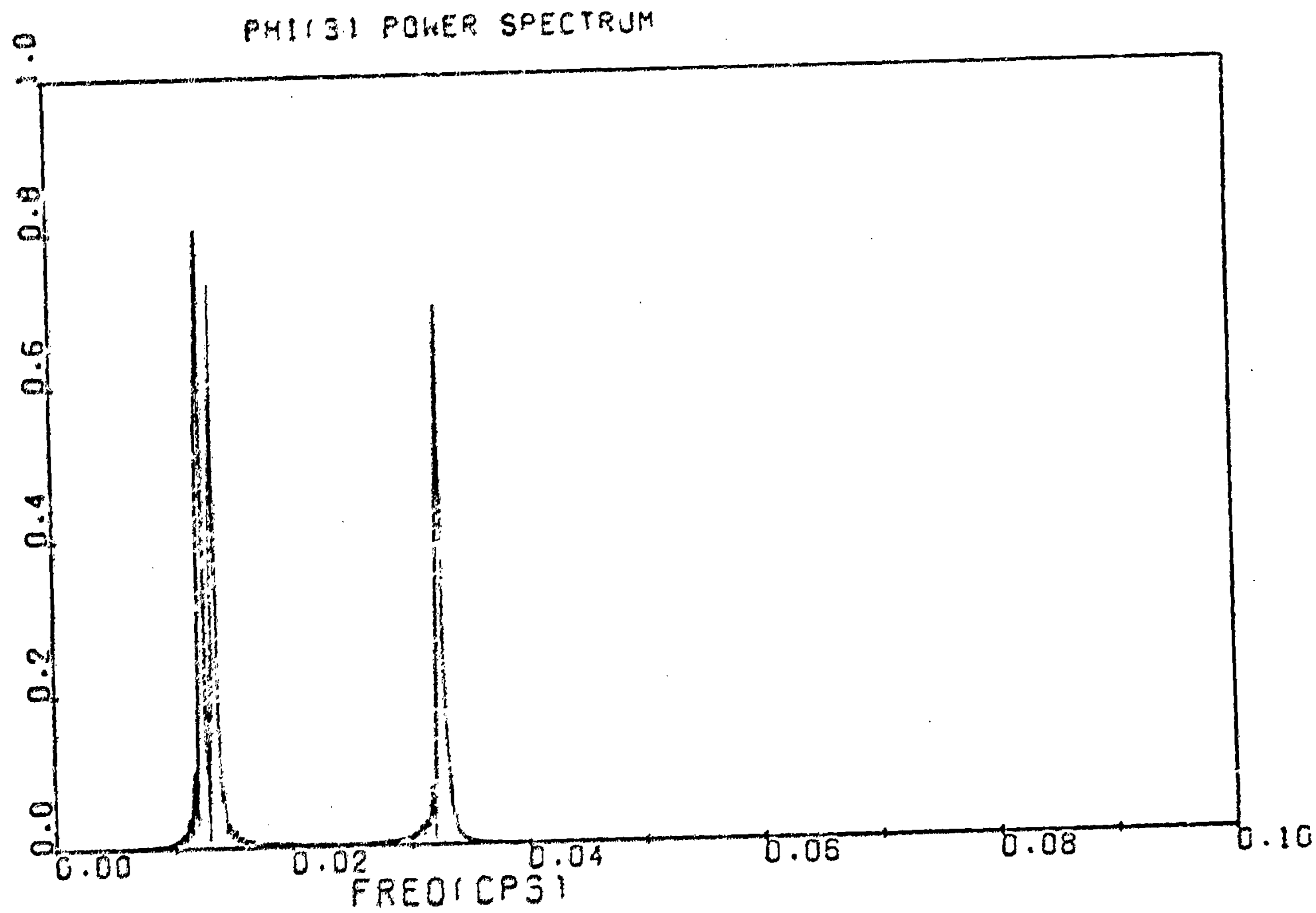


Figure 15(b). Power spectrum of boom 3 oscillations.

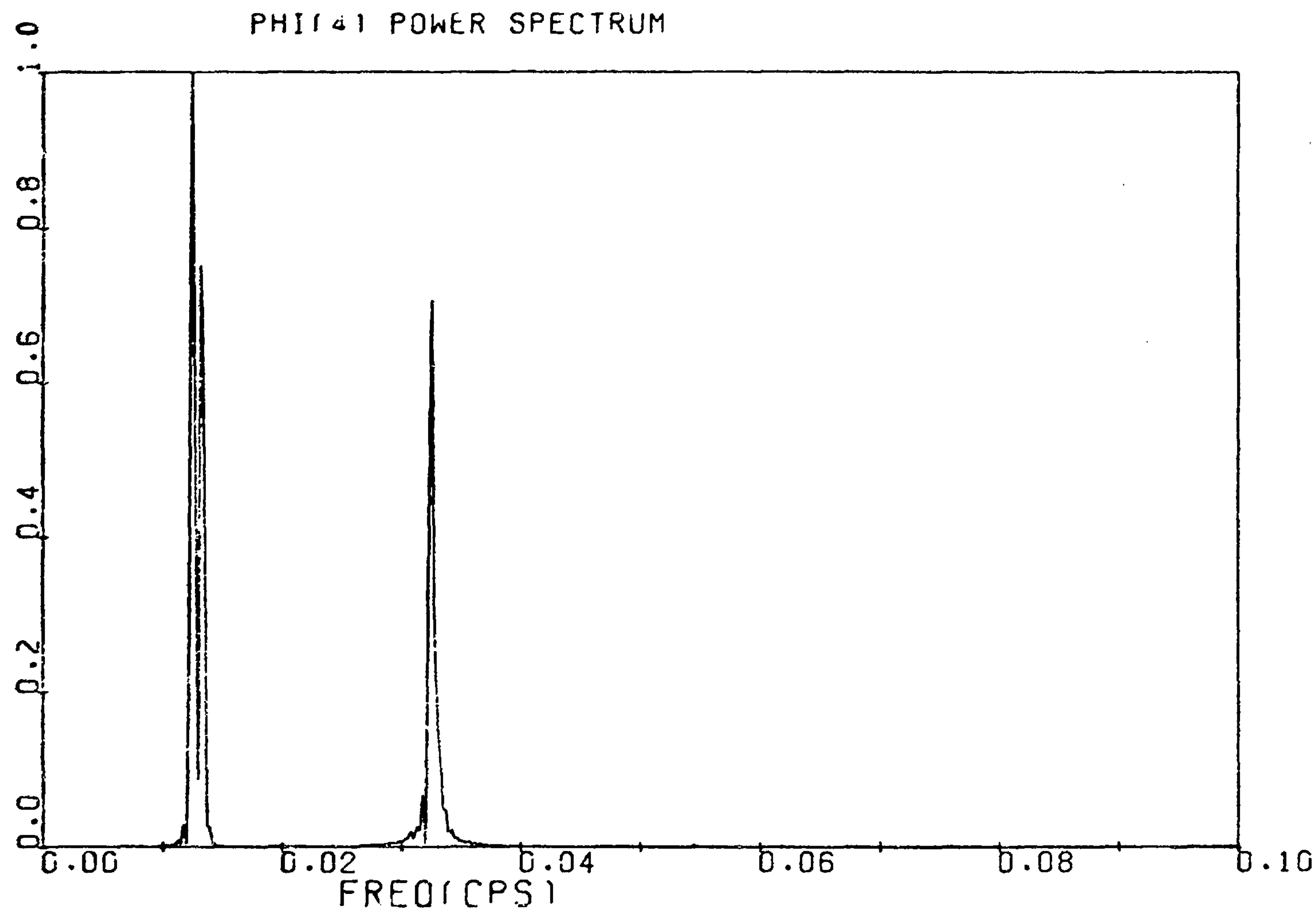


Figure 15(i). Power Spectrum of boom 4 oscillations.

The third set of simulation graphics depicts the effect of boom deployment [Fig. 16 (a-d)]. All boom lengths are 40 ft. initially and 50 ft. finally. The modes are excited by deployment (period τ = 200 sec.) and by initial boom deflections (-.09, -.07, -.09, -.07 radians). During the deployment period, the hub spin slows down due to conservation of angular momentum [Fig. 16 (a)]. Oscillation of hub spin rate indicates the presence of coupled mode. Translational oscillation is not involved because of the symmetry of the booms. A plot [Fig. 16(b)] of deflections of booms 1, or 3, and 2, or 4 (curve with asterisks) shows particularly large deflection amplitudes during deployment period in which forced oscillation (c.f. Chapter 2) are present. If the initial boom deflections were identical, simultaneous deployment/retraction for all booms would excite the coupled mode only. Since the initial boom deflections are not really identical, an uncoupled mode is present, as revealed by the power spectra of boom 1 (or 3), and boom 2 (or 4) in Fig. 16(c) and Fig. 16 (d), respectively. However, the coupled mode is dominant. There is a small bump (or tail) attaching to the right hand side of each of the sharp spectral lines. This phenomenon is due to the higher spin rate (and hence higher mode frequency) before completion of deployment (and hence slow-down of spin). The bump near 0 hertz is due to the nearly constant force step function during deployment. Digital print-outs show that the minimum of this bump is at 0.005 c.p.s (inverse of 200 sec., the deployment period τ).

To illustrate the point that the amplitude of boom oscillation after deployment/retraction depends on when the deployment/retraction stops (see point #7 on pg. 109), a plot is presented [Fig. 17] in which deployment stops 10 sec. (i.e. 1/20th of τ) earlier than in Fig. 16 (b), and the booms swing to the other side in large amplitudes, just after deployment stops.

Figure 16 (a-d)

Simulation graphics: Set #3
Satellite boom lengths: 40 ft. (all) initially,
50 ft. (all) finally

Deployment period: 200 sec.
Initial boom deflections: (-.09, -.07, -.09, -.07) radians

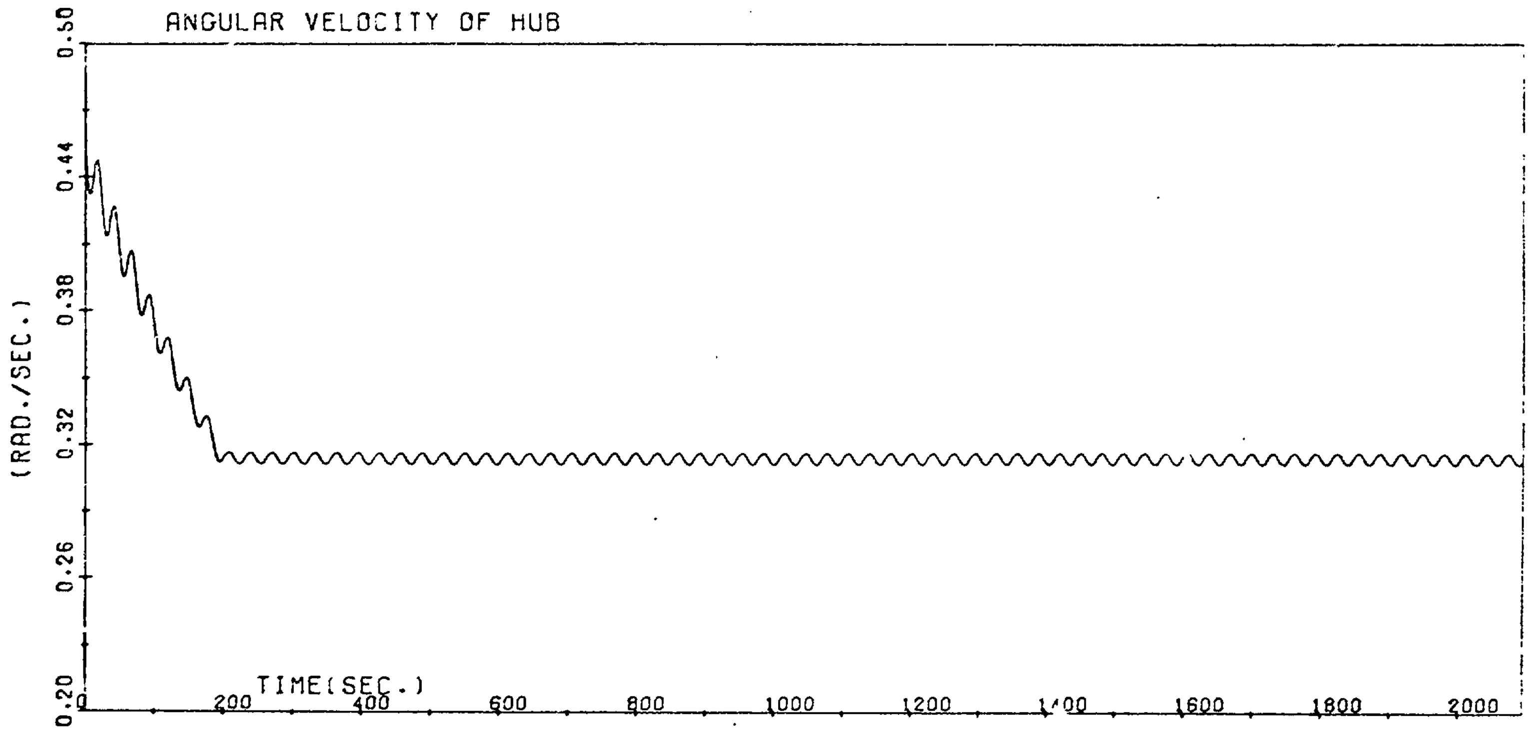


Figure 16(a). Boom deployment slows down the spin of the hub, due to conservation of angular momentum.

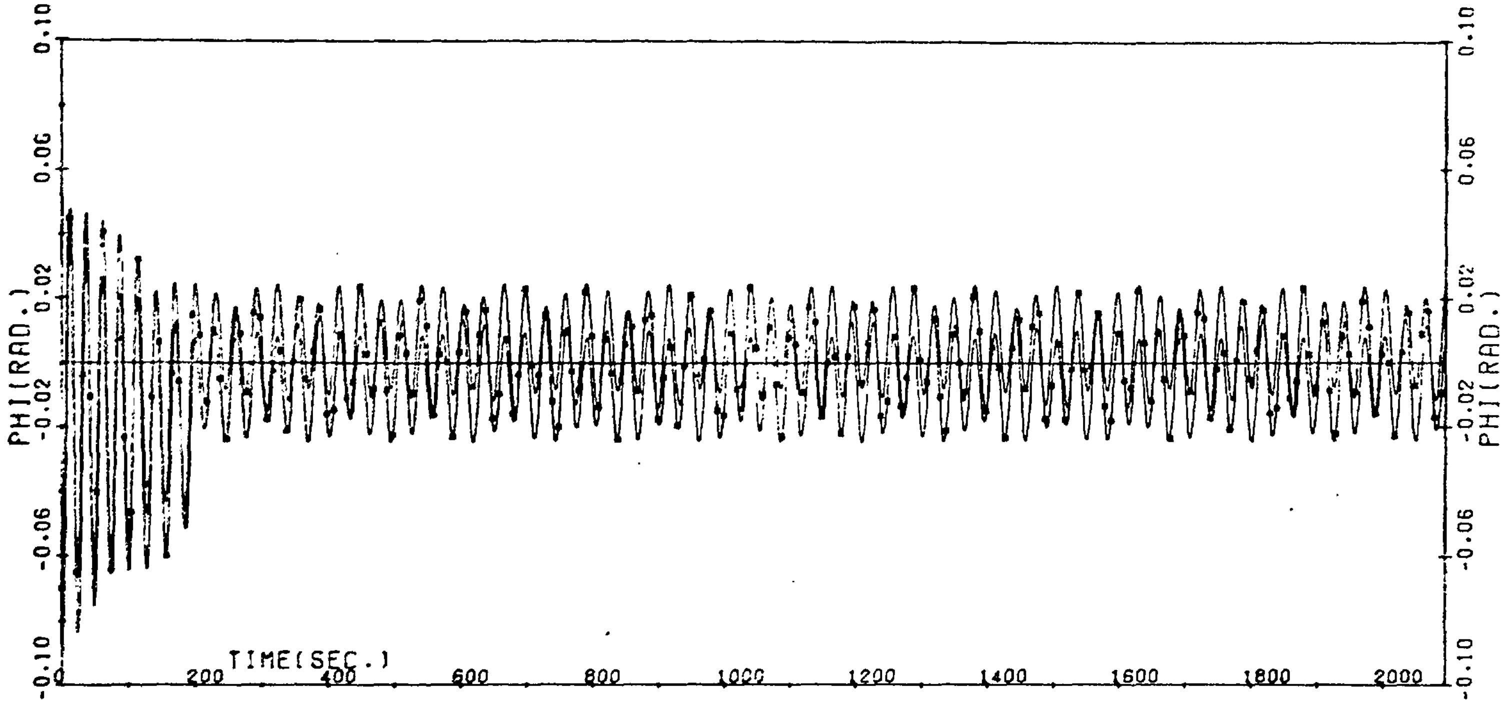


Figure 16(b). Deflections of boom 1 (or 3) and 2 (or 4) with asterisks show particularly large amplitudes during deployment period (t≲200 sec).

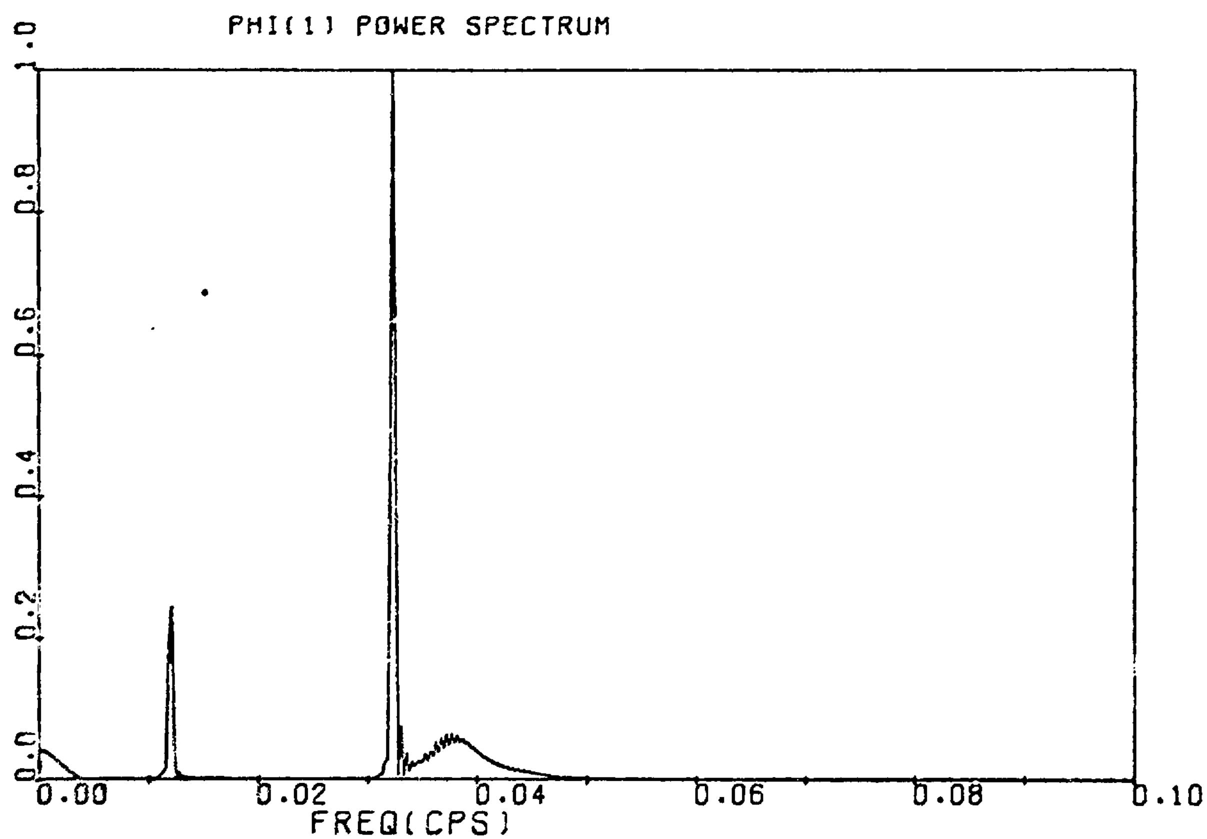


Figure 16(c). Power spectrum of boom 1 (or 3) oscillations. Simultaneous deployment or retraction of all booms enhances the coupled mode. The tails on the higher frequency side are due to higher spin rate before completion of deployment. The very low frequency bump is due to the force caused by deployment during $t \lesssim \tau$ (200 sec.). Digital print out shows that minimum of this bump is at 0.005 c.p.s. (inverse of τ).

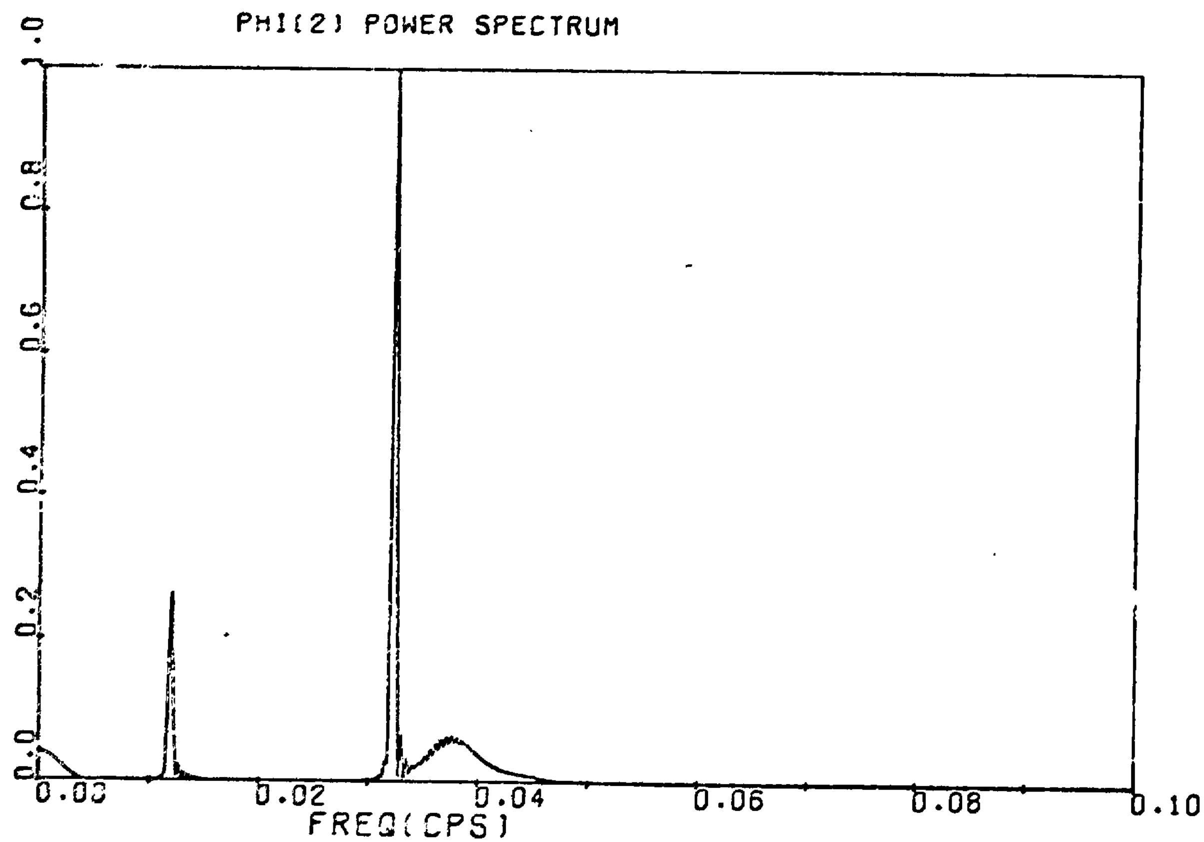


Figure 16(d). Power spectrum of boom 2 (or 4) oscillations. See caption on the preceding figure.

The next set of simulation graphics (set #4) [Fig. 18 (a-i)] shows the combined effects of unsymmetrical boom retraction, hub translation, and unequal boom lengths. Initially, all boom lengths are equal (45 ft.), and finally, their lengths are 35, 45, 45, 45 ft. Retraction period is 200 sec. and the total simulation time is 4100 sec. furnishing 2^{12} data points at 1 sec. intervals for fine spectral resolution ($\Delta\omega$ = 0.000122c.p.s) in the subsequent Fourier analysis. During the retraction period, the hub spin increases [Fig. 18 (a)] in contrast to the deployment case. Simulations of translational variables ($X = X_o + X$, $Y = Y_o + Y$) show that the hub center oscillates about the center of mass of the satellite system, and no runaway occurs [Fig. 18(b, c)]. The amplitude of oscillation of the retracting boom [Fig. 18 (d)] is very large due to forced oscillation during retraction period (see Section 2.4)]. The non-retracting booms 2 (with asterisks in Fig. 18d), 3 and 4 (with astrtisks, Fig. 18e) move in opposite directions relative to the retracting boom. Beats are prominant in boom oscillations [Fig. 18 (d, e)], resulting in the splitting of spectral lines [Fig.18 (f-i)]. The analysis in Chapter 5 (Equ. 5-12) has predicted a two pronged splitting of the uncoupled mode frequency for pairwise equal boom cases with translations. However, minor complications in the splitting would occur if the boom lengths are unbalanced [Fig. 18 (f-i)]. The coupled mode and higher harmonics frequencies are too weak to show up. A bump at very low frequency in the power spectrum of the retracted boom [Fig. 18 (f)] is due to the 200 sec. constant retraction resulting in an almost constant force during the retraction period. The minimum of this bump and, in fact, all four bumps can be identified at 0.005 c.p.s. (inverse of 200 sec.) and again at 0.01 c.p.s. (which is n/τ, where $n = 2$, $\tau = 200$). Small local maxima and minima can be seen on the bump (and, in fact, they are everywhere); they are due to the use of finite length (4096 sec.) of time series. The location of the first local minimum is at 0.000244 c.p.s. (inverse of 4096 sec.). The bump of boom 3 is very weak but can be revealed by means of digital printouts [not shown].

Very short boom behavior is simulated in the next figure [Fig. 19]. All boom lengths are equal to 9.5 ft. initially, and 1.5 ft. finally. Retraction period is 160 sec. The wavelength of a boom decreases as the boom is being retracted, and the amplitude rapidly grows to very large value (it would grow to infinity at zero boom length). Strong damping (not due to Coulomb damper) is prominent for short length booms (see Section 2.4). Since all booms behave identically in this simulation, it is sufficient to show the behavior of only one boom.

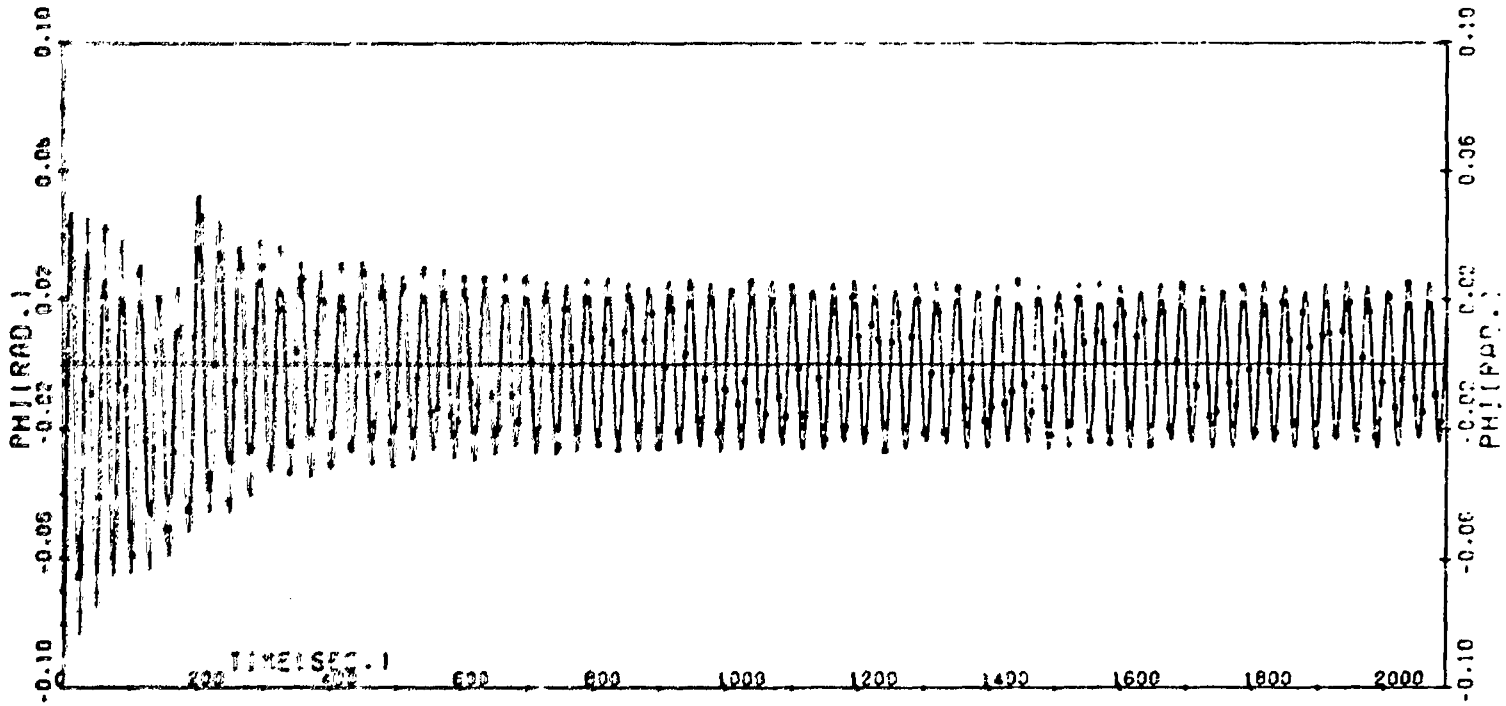


Figure 17. Deployment stops earlier (by $\tau/20$) than in Fig. 9.8(b) illustrating that the amplitude of boom oscillations after deployment depends on when deployment stops

Figure 18 (a-i)

Simulation graphics: Set #4
Satellite boom lengths: 45 ft. (all)initially,
35, 45, 45, 45 ft. finally
Retraction period: 200 sec.
Initial boom deflections: Nil

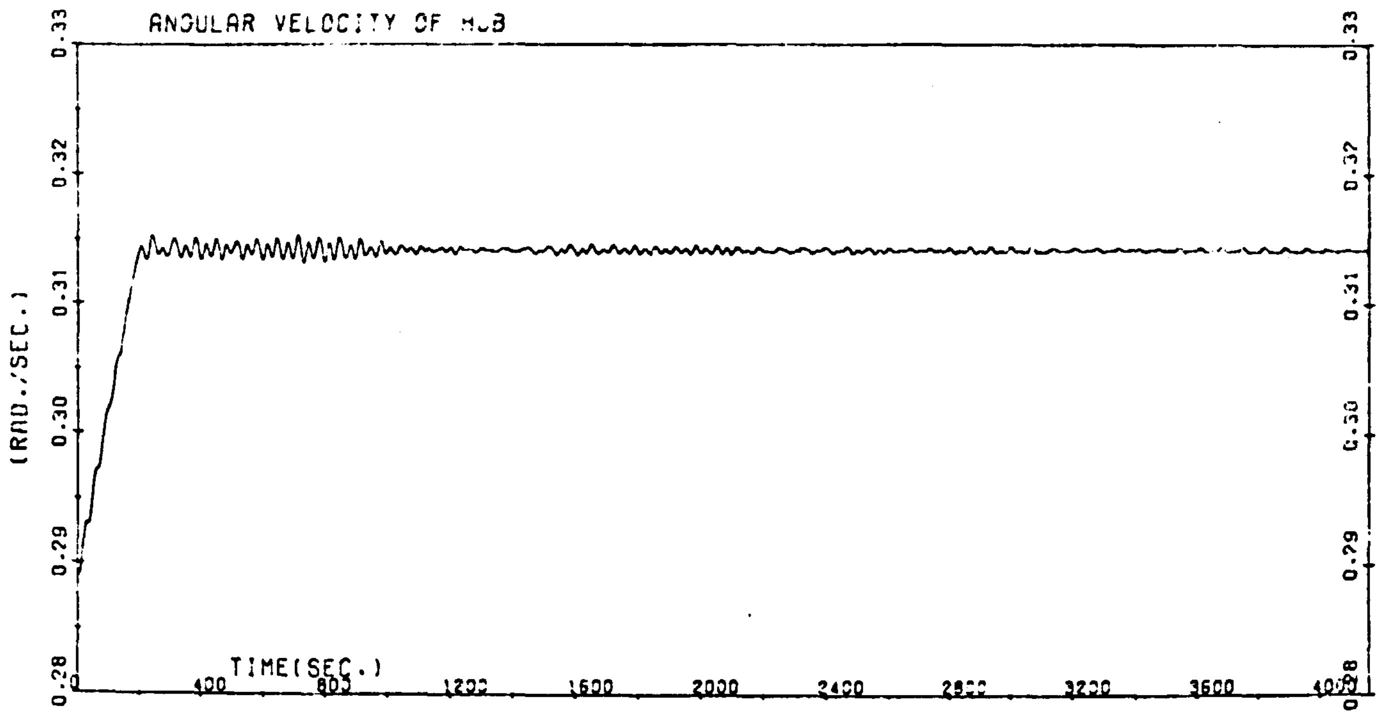


Figure 18(a). Hub spin rate increases during boom retraction.

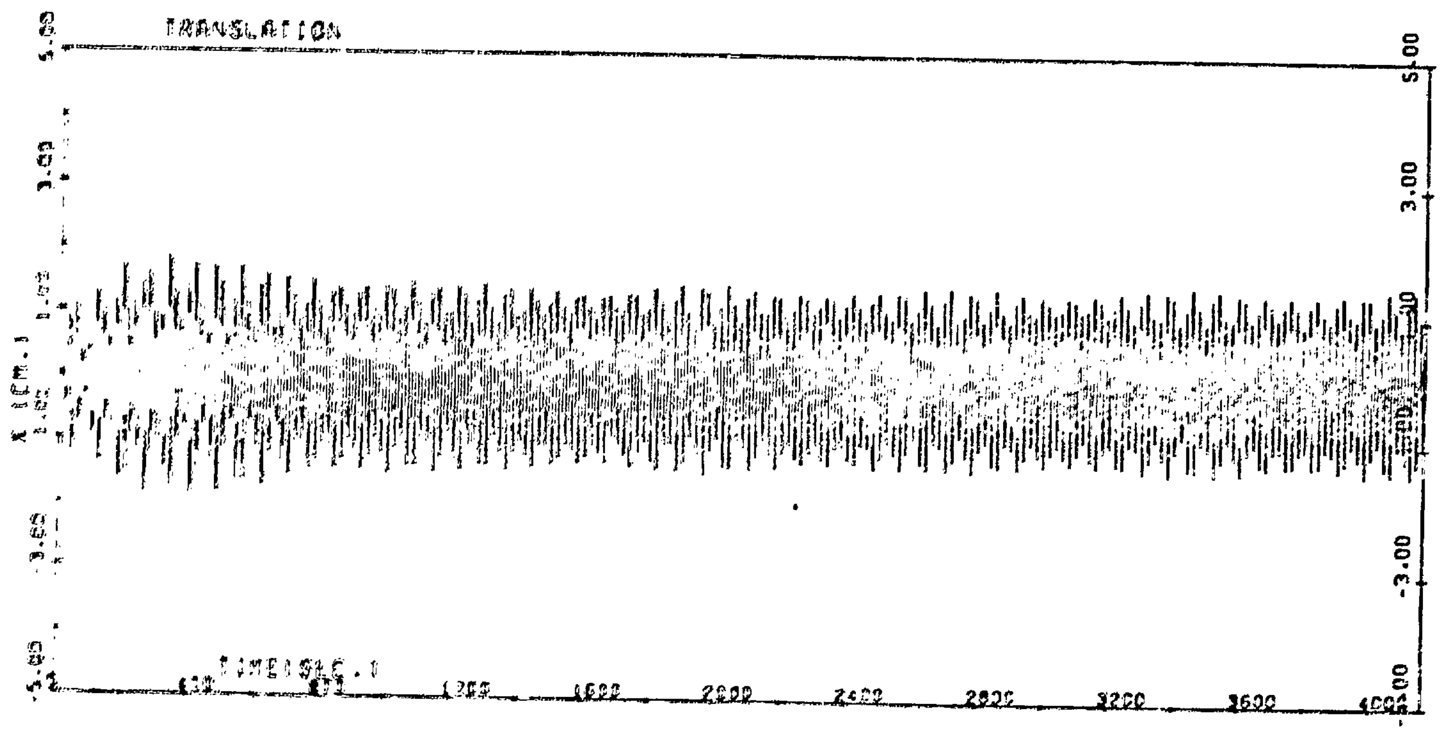


Figure 18(b). Translation ($X = X_o + X'$) of the hub center, oscillating around the center of mass of the satellite system.

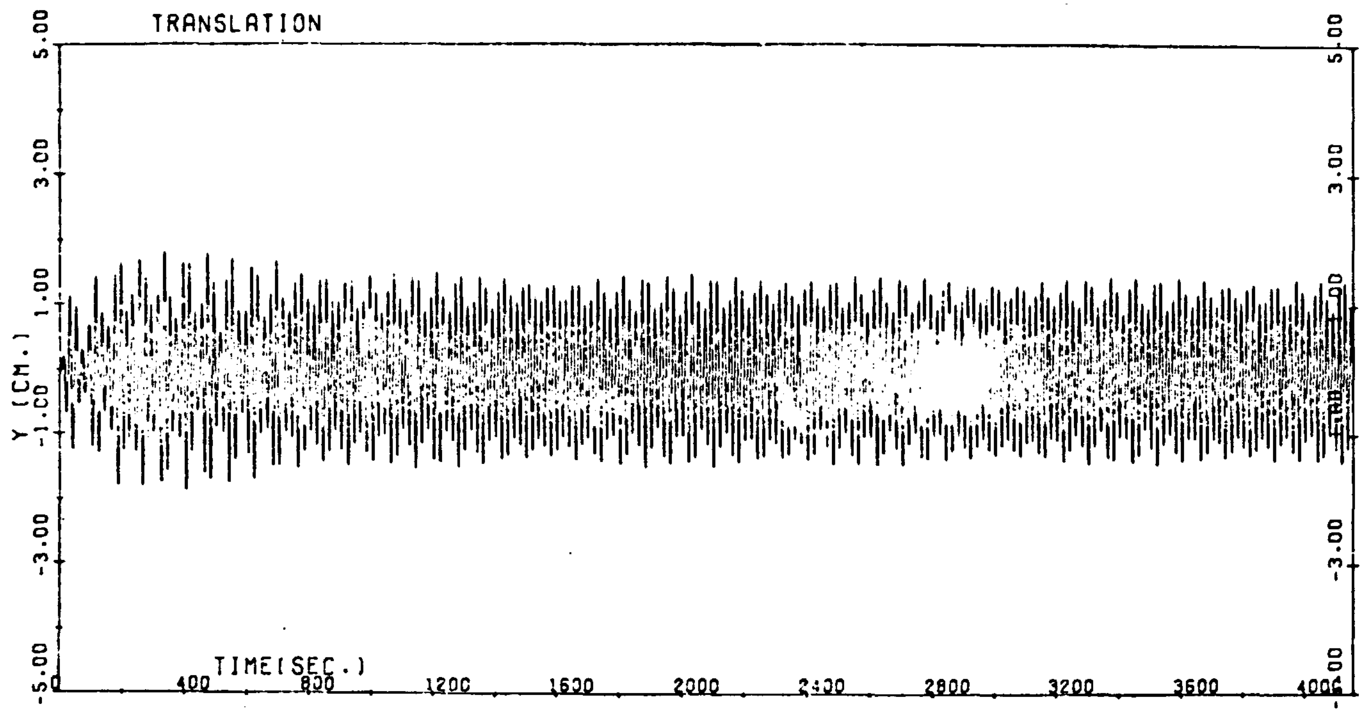


Figure 18(c). Translation ($Y = Y_o + Y'$) of the hub center, oscillating around the center of mass of the satellite system.

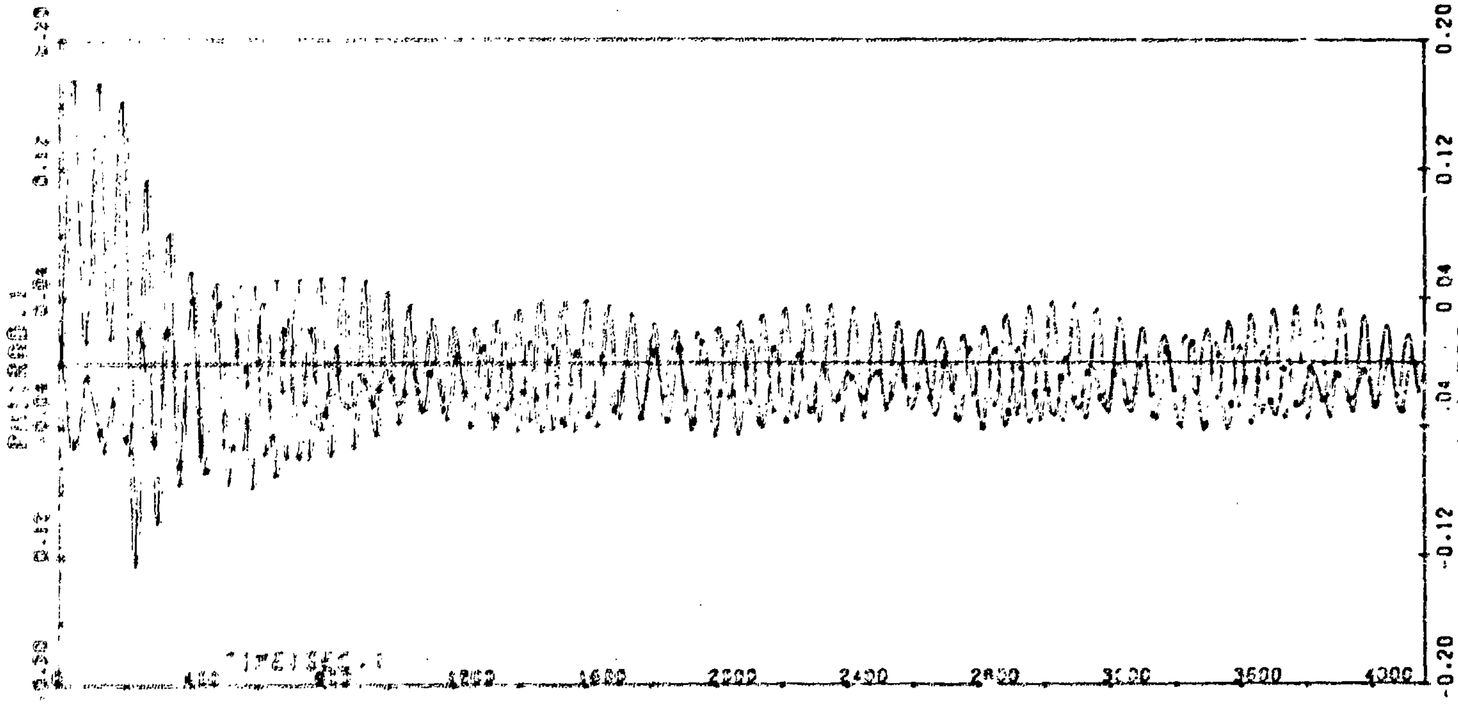


Figure 13(d). Deflections of boom 1 and boom 2 (with asterisks). The amplitude of deflections of the retracting boom is large, and it is completely out-of-phase with those of non-retracting booms (2, 3, 4). Beats can be clearly seen.

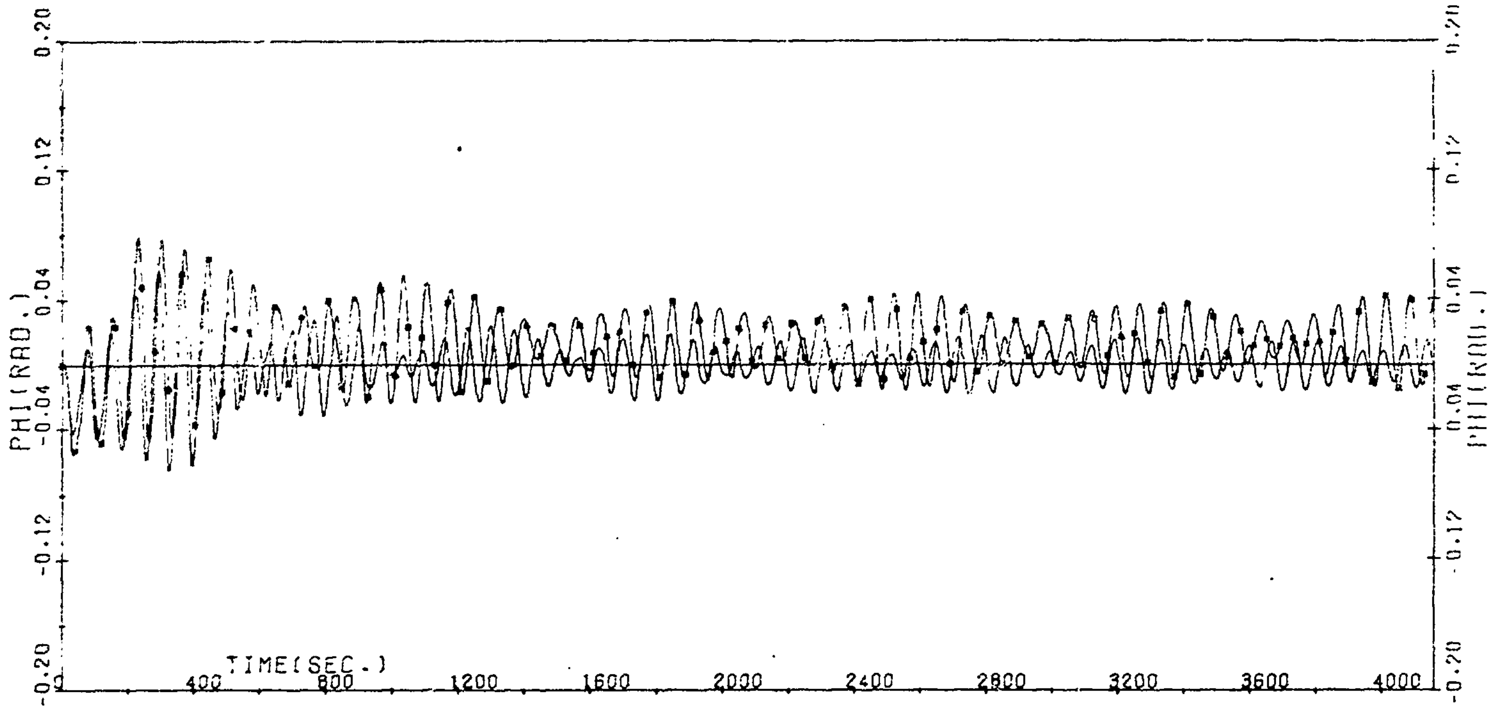


Figure 18(e). Deflections of boom 3 and boom 4 (with asterisks). Beats can be clearly seen.

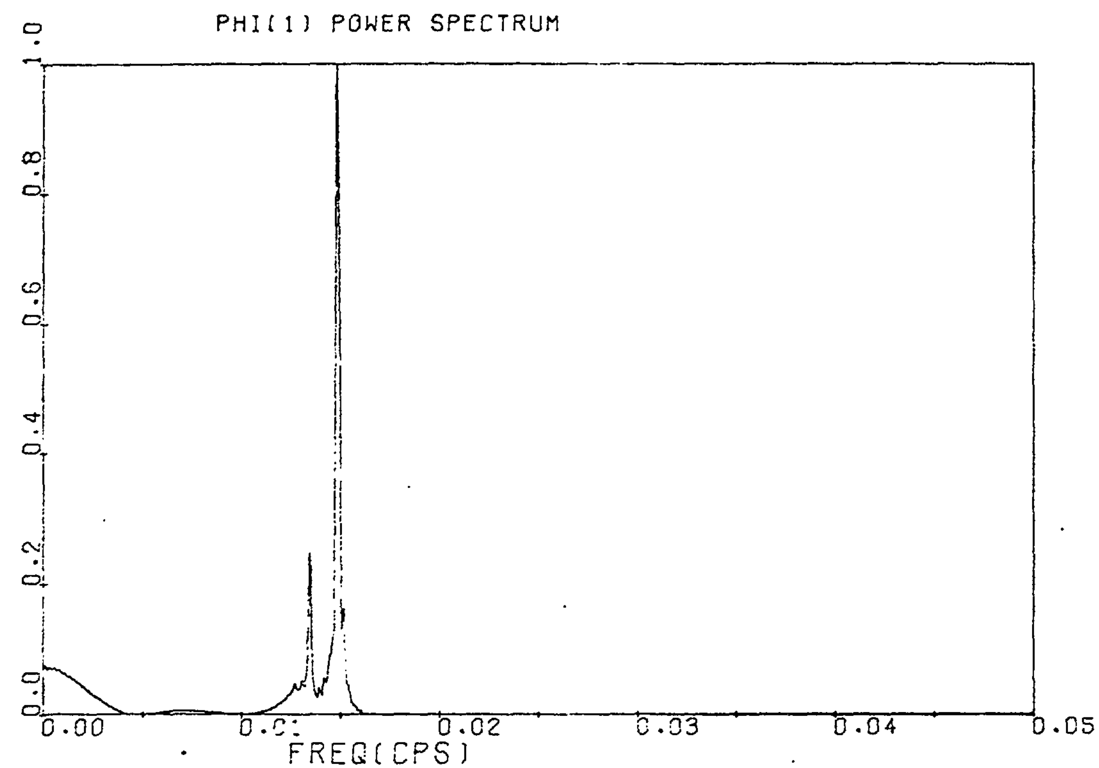


Figure 18(f). Power spectrum of boom 1 oscillations. The splitting of the spectral line is due to the translation induced beats in boom deflections. The low frequency bump is due to constant deployment for 200 sec. Minima of the bump can be clearly seen at 0.005 and 0.01 c.p.s. (i.e. n/200 c.p.s., n = 1, 2).

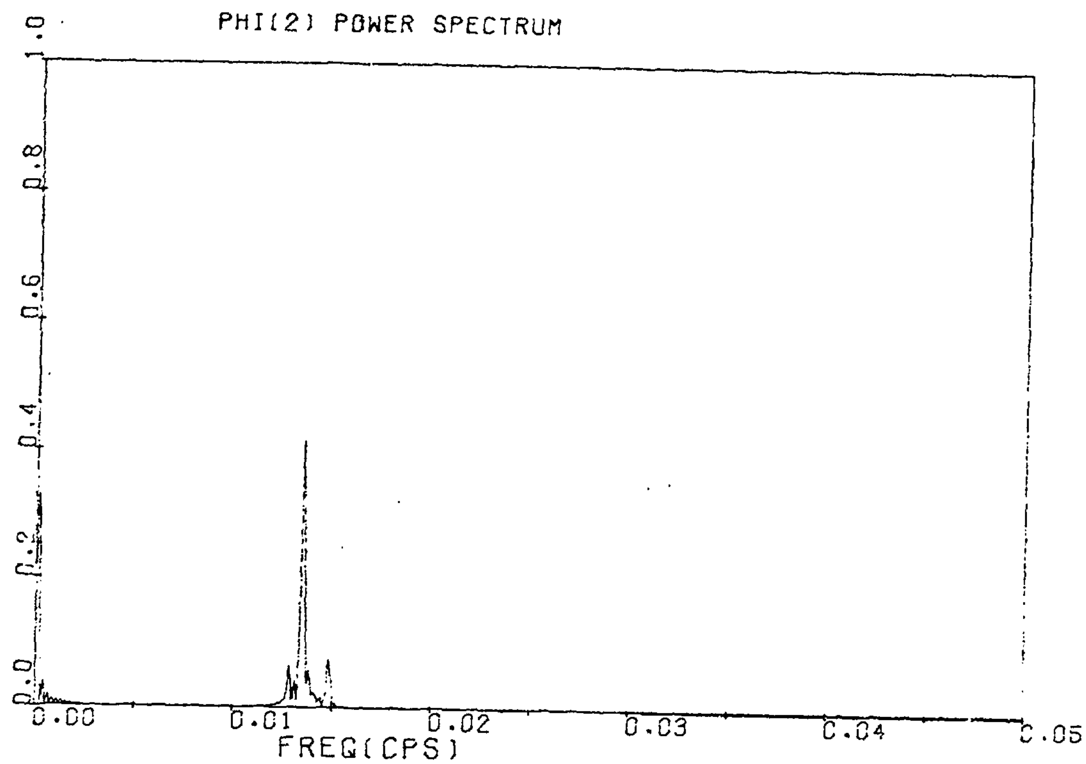


Figure 18(g). Power spectrum of boom 2 oscillations. Many small spurious minima are present due to the finite length of wave train (4096 sec.)

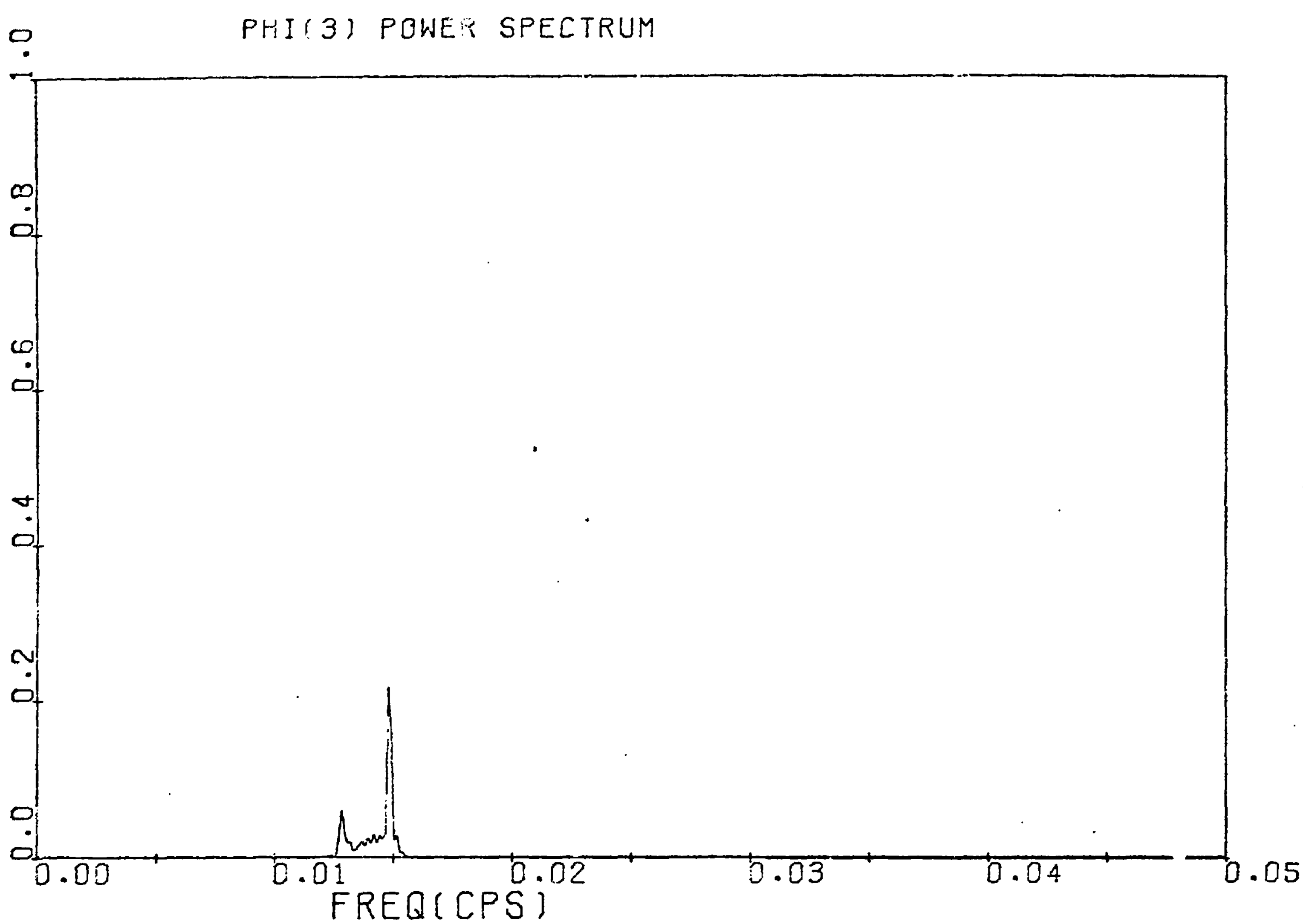


Figure 18(h). Power spectrum of boom 3 oscillations. The low frequency is too weak to show up graphically.

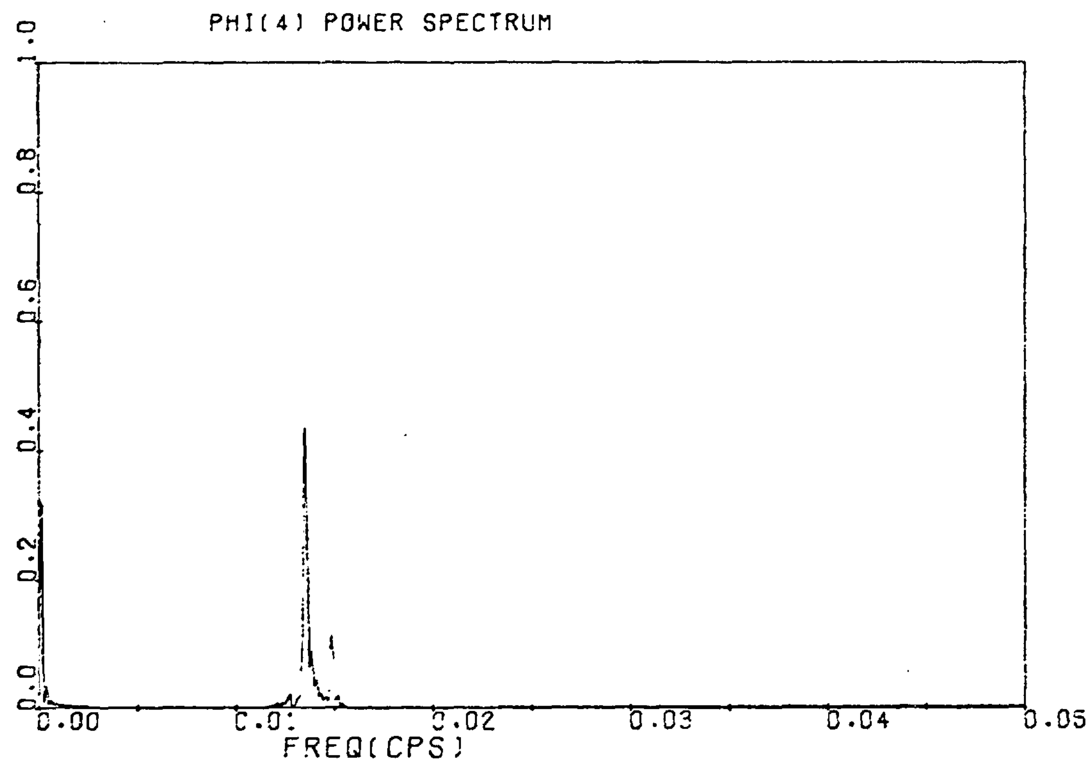


Figure 18(i). Power spectrum of boom 4 oscillations.

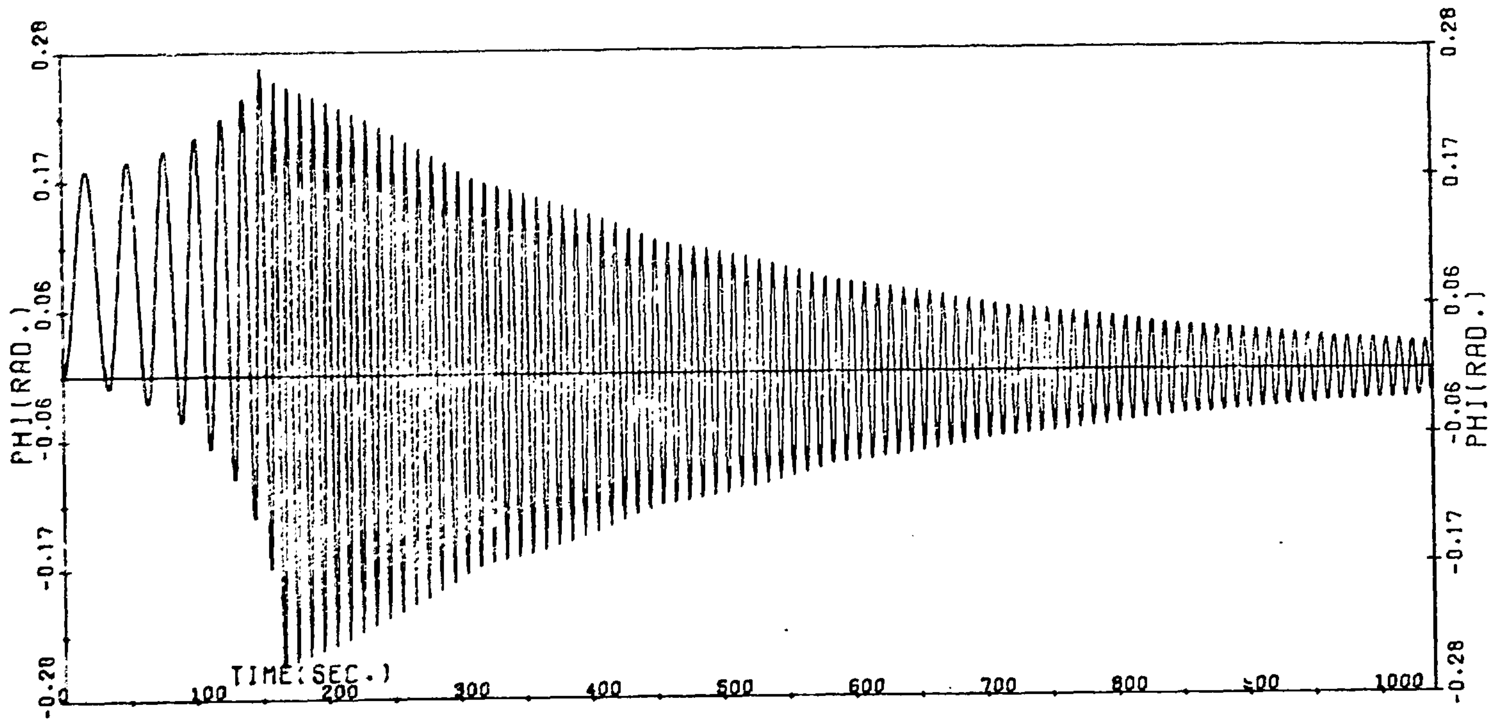


Figure 19. Very short boom deflections showing rapidly increasing amplitude during retraction (all 9.5 ft. to all 1.5 ft. in 160 sec.), and strong damping after retraction. The pure coupled mode is excited with all booms in phase.

The fifth set of simulation graphics shows the effect due to the deployment of a pair of booms (boom 1 and its opposite boom 3) and the simultaneous retraction of the other pair of booms (2 and 4) [Fig. 22 (a-d)]. All boom length are 20 ft initially, and 30, 10, 30, 10 ft. finally. Retraction period is 200 sec. and the deployment/retraction rates are equal and opposite to each other (viz., .05 ft. /sec.). This maneuver of wire booms causes a net increase in moment of inertia of the system and therefore a slow down of satellite spin results [Fig. 22 (a)]. No translation occurs because of the symmetry of the system throughout the maneuver. Deflections of the retracting booms are in opposite direction to the deploying booms (asterisked curve) [Fig. 22 (b)]. Amplitudes of the retracting booms are larger than those of the deploying ones, since oscillations of retracing booms are less stable (see Paragraph (4) of Chapter 9). All boom amplitudes during deployment/retraction are somewhat large, because the magnitude of the relative deployment/retraction rate is twice that in a pure deployment or retraction case. The power spectra of boom oscillations show the presence of both partially coupled and totally coupled modes [Fig. 22 (c,d,)] (compare the harmonic frequency results in Figure 21). The low frequency contributions in the power spectra are again due to the 200 sec. of forced oscillations caused by deployment/retraction of wire booms.

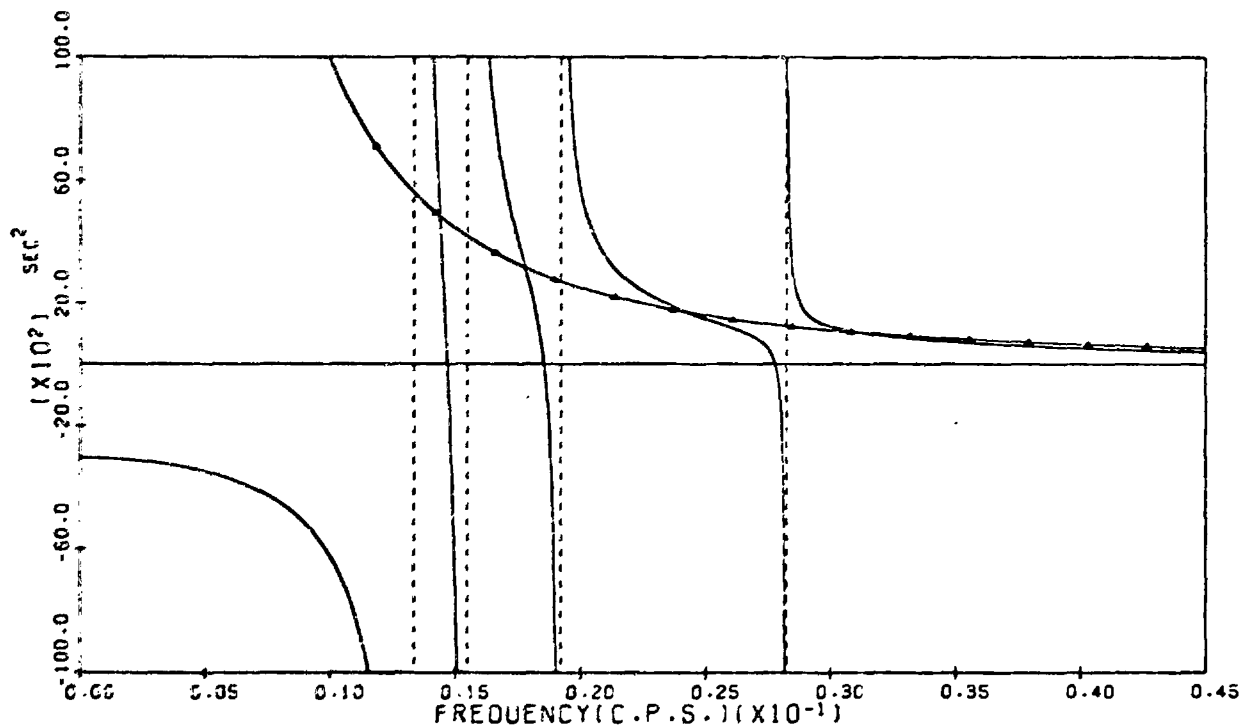


Figure 20. Computer generated graphical solutions of the analytic formula [Equ. (7-12)] for the harmonic frequencies of 1975 satellite system with boom lengths 10, 20, 30, 40 ft. The frequencies are given by the four intersections of the curves (c.f. Fig. 7).

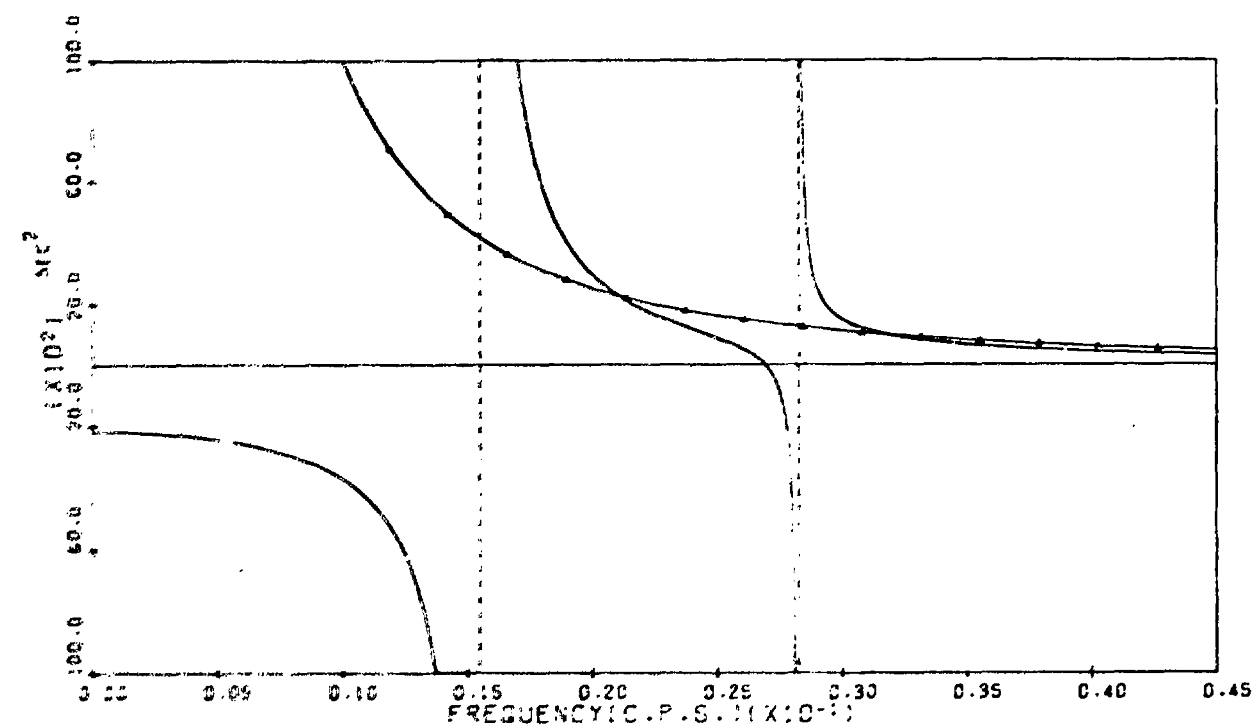


Figure 21. Computer generated graphical solutions of the analytic formula [Equ. (7-10)] for the harmonic frequencies of 1975 satellite system with boom lengths 10, 30, 10, 30 ft.; (these are the same boom lengths used in simulation set #5. See Figure 22(a-d). The frequencies are given by the two vertical asymptotes (for the uncoupled modes) and the two curve intersections (for the partially coupled and fully coupled modes) (c.f. Figure 5).

Figure 22 (a-d)

Simulation graphics: Set #5
Satellite boom lengths: 20 ft. (all) initially,
30, 10, 30, 10 ft. finally
Deployment period: 200 sec.
Initial Boom Deflections: Nil

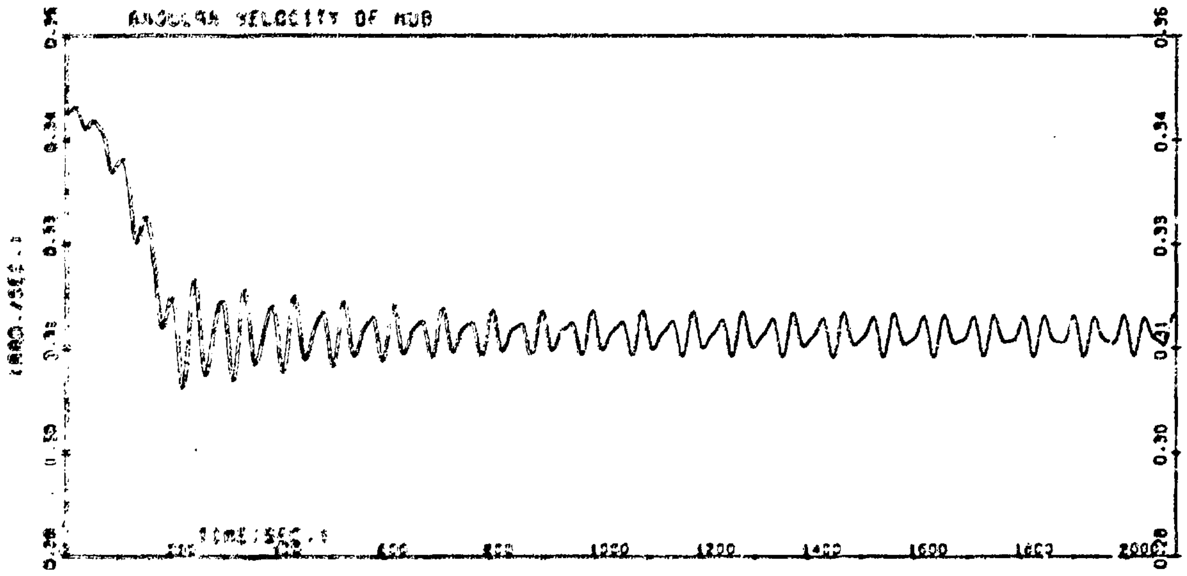


Figure 22(a). Deployment of one pair of booms and the simultaneous retraction of another pair for the same lengths give a net increase in moment of inertia slowing down the hub spin.

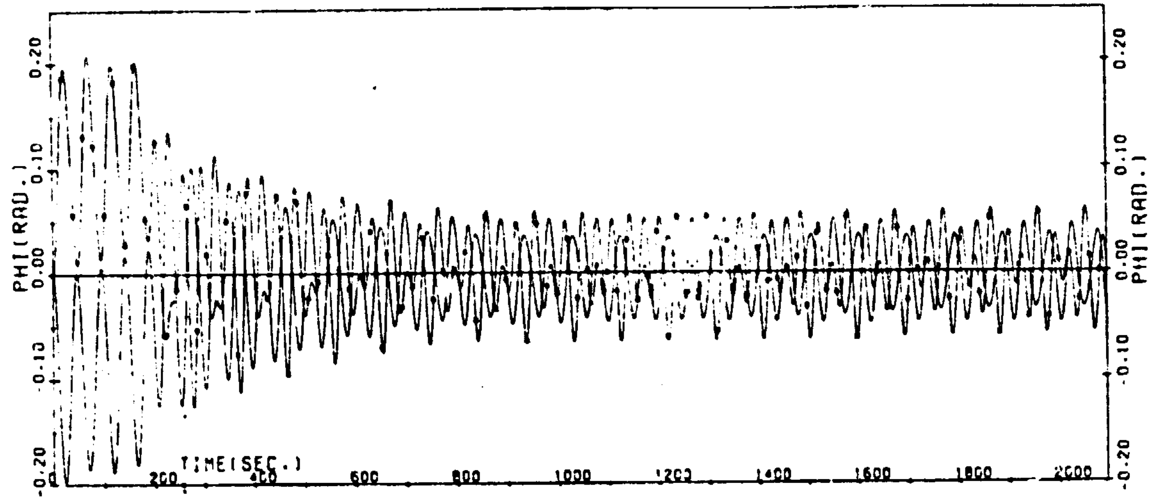


Figure 22(b). Deflections of the retracting booms 1 and 3 are in opposite direction to the deploying booms 2 and 4 (with asterisks). Deflection angles are large because the magnitude of the relative deployment/retraction rate is twice that in previous simulation sets.

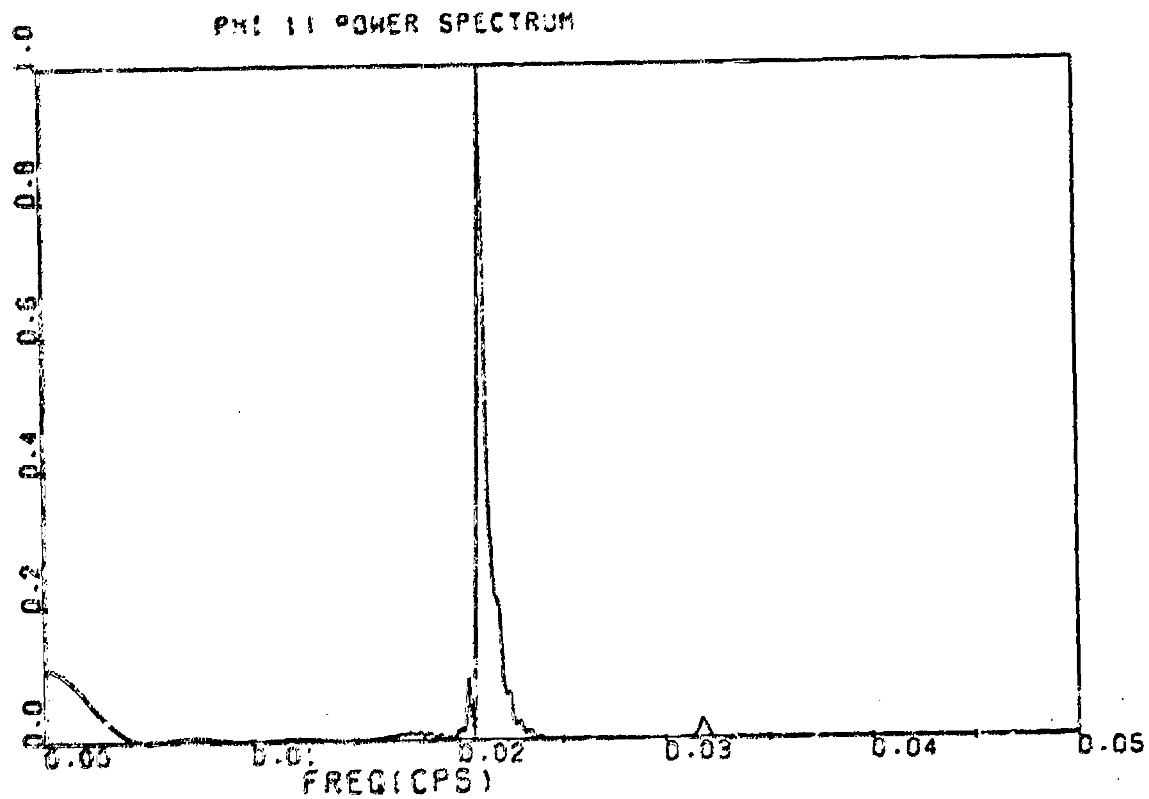


Figure 22(c). Power spectrum of boom 1 (or 3) oscillations shows the presence of both partially coupled and totally coupled modes. The low frequency bump is again due to 200 sec. of forced oscillation.

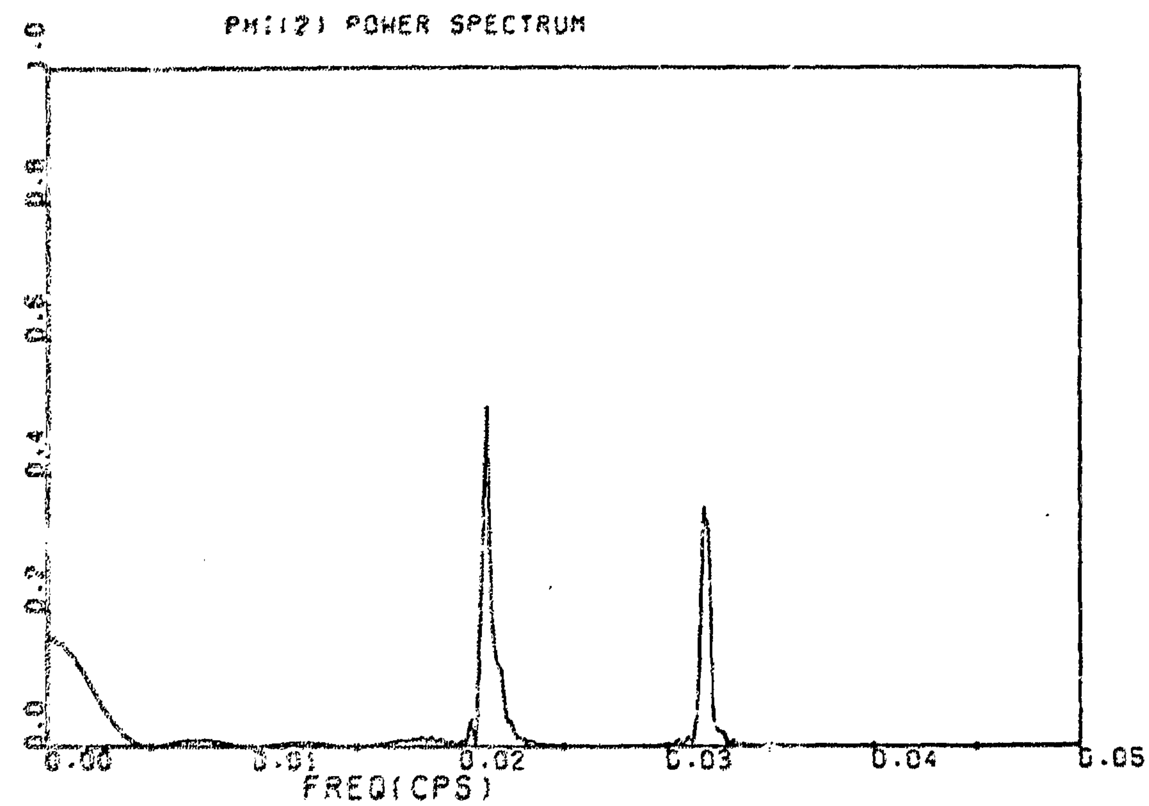


Figure 22(d). Power spectrum of boom 2 (or 4) oscillations shows the presence of both coup and uncoupled modes. The low frequency contributio again due to 200 sec. of forced oscillations.

The last set of simulation graphics (Set #6) is to demonstrate the harmonic oscillations out of the spin plane of the system. Such oscillations are expected to be insignificant during actual satellite experiment because of the installation of a wobble damper, which is considered to be very effective (see Section 3.7, ref. 1). Hence, there is no need for elaborate simulations for out-of-plane oscillations. However, for the purpose of gaining insights, it is useful to perform a simulation [Computer program SYNHARM] in harmonic approximation without damping. The method of Fourier synthesis is most appropriate since all the harmonic variables have been calculated (Chapter 6). For a given set of harmonic variables, the time evolution of the variables are simulated. In Set #6 the initial values (ψ_1, ψ_2, ψ_3, ψ_4, θ_1, θ_2, Z) are given as (-.01008, .04140, .05160, .04164, .01422, -.2844, .01198) where the angular variables (ψ_i, θ_i) are in radians and the translations in feet. The time series of ψ_1 and ψ_2 (asterisked) and those of ψ_3 and ψ_4 (asterisked) are displayed in Figure 23a and b respectively. All four nontrivial modes are present. The coupled modes can be clearly seen in θ_1 and θ_2 (asterisked) time series (Figure 23[c]), where the coupled mode involving booms 2 and 4 is twice in magnitude compared to that involving the other pair. The Z - motion is due to the presence of jelly-fish mode, and since it is a pure sinusoidal wave of small amplitude, its time series is not of much interest and thus not shown. The spin axis sweeps a conic in space. A cross-section of the conic in this simulation shows a clover leaf pattern [Fig. 23 (d)]. The conic sweep starts in the third quadrant, moves into the fourth, then second, third quadrants, etc. and is found moving in the second quadrant at the end of a 24 sec. simulation run.

Figure 23 (a-d)

Simulation graphics: Set #6
Satellite boom lengths: 45 ft. (all)
Deployment period: Nil
Initial Boom Deflections: -.01008, .04140, .05160, .04164 radians
Initial Hub Inclinations: -.01422, -.02844 radians
Initial Z- position: .01198 ft.

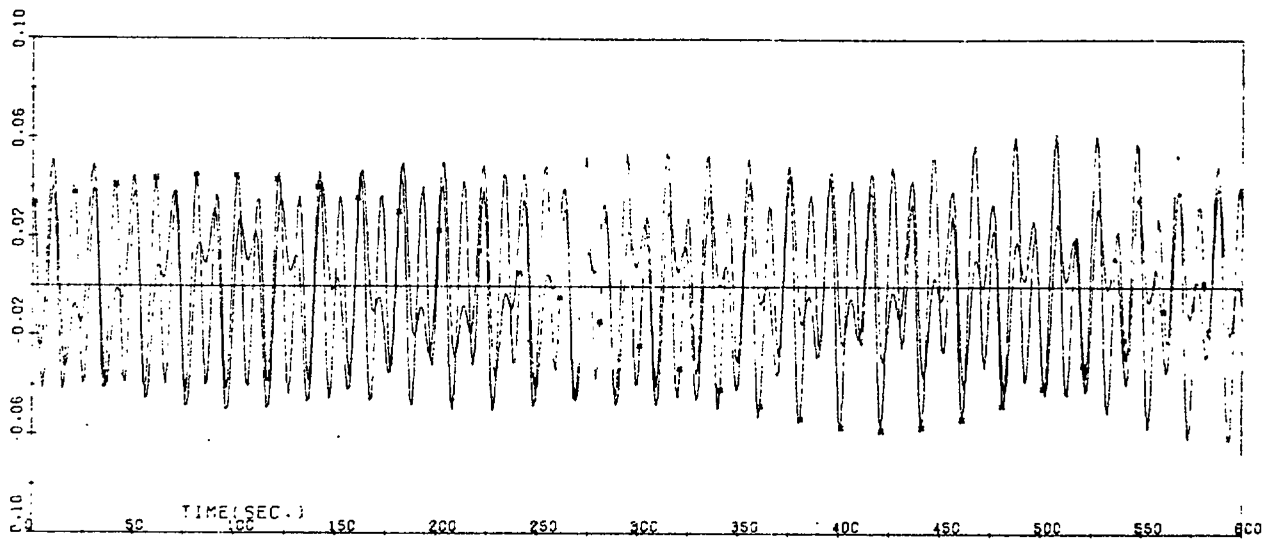


Figure 23(a). Time series of out-of-spin-plane angular deflection ψ_1 and ψ_2 (asterisked) of two adjacent booms. The sinusoidal appearance of the asterisks is due to absence of damping in this simulation.

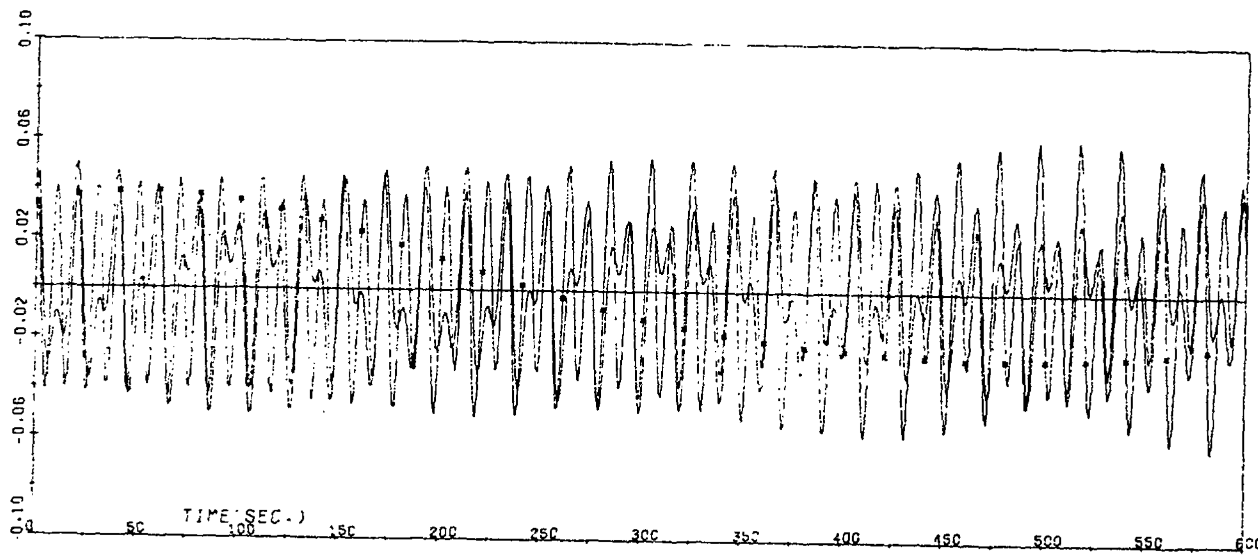


Figure 23(b). Time series of out-of-spin-plane angular deflections ψ_3 and ψ_4 (asterisked) of two adjacent booms. The sinusoidal appearance of the asterisks is due to the absence of damping in this simulation.

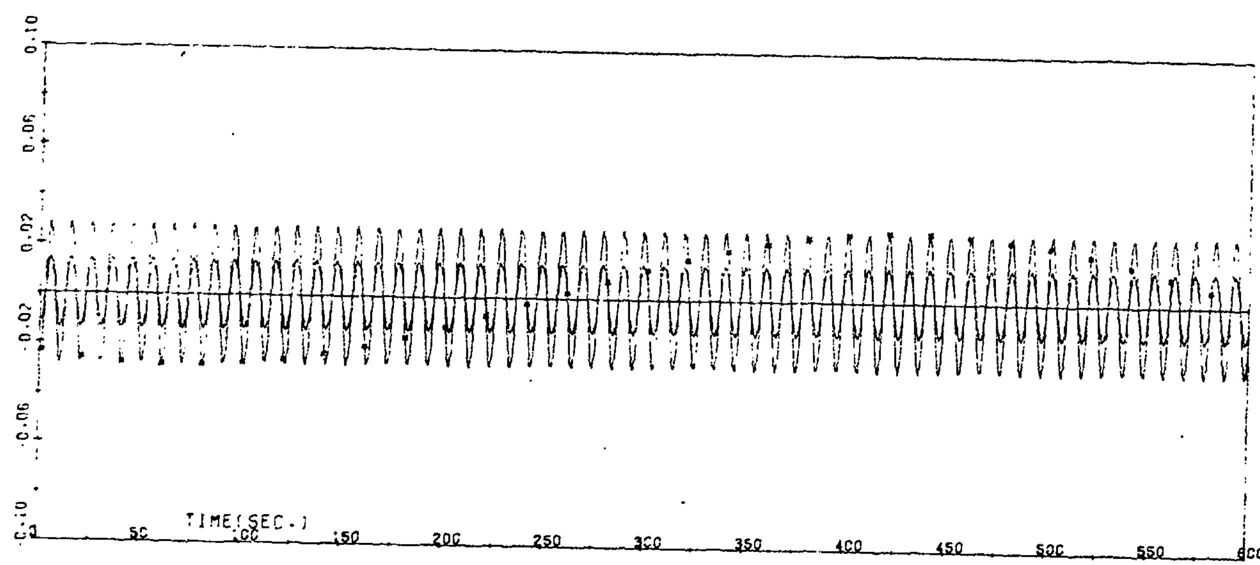


Figure 23(c). Time series of out-of-spin-plane inclinations θ_1 and θ_2 (asterisked) of hub. The harmonic mode involving θ_2 (and therefore booms 2 and 4) is twice in magnitude compared to that involving θ_1 and the other boom pair.

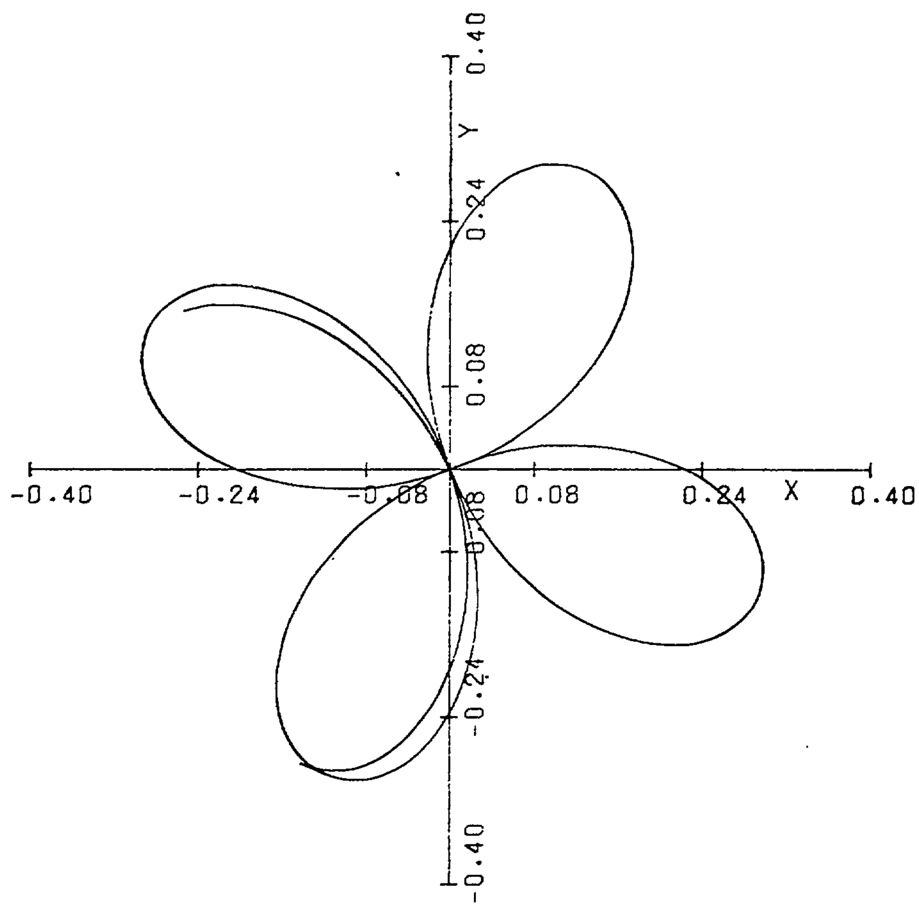


Figure 23(d). Conic cross-section swept by the spin axis precessing around a spatially fixed Z-axis in a finite time span. The scale represents in feet the pattern traced by the spin axis on a plane 10 ft. above the center of mass. The sweep starts in the third quadrant and ends in the second one in 24 sec.

CHAPTER 10

PROGRAM LISTINGS, INPUT FORMATS AND SAMPLE OUTPUT

10.1 Introduction

For full simulation of the satellite dynamics including built in Coulomb damper actions, large angular deflections leading to possible higher harmonics, combinations of boom deployment/retractions, and possible non-linear mode-mode coupling, Lagrangian equations of motion should be used. They are found to be a set of coupled second order non-linear differential equations (Chapter 3). The time-dependent solutions of these coupled differential equations, which embody all the dynamics of the system, represent the entire history, or simulation, of the behavior of the various components of the satellite-wire boom system. For a specific simulation, controlled initial conditions may be imposed.

10.2 Method of Simulation

Hamming's modified predictor-corrector method [12] is chosen to integrate the differential equations by virtue of its relative stability and accuracy. It has the advantage of self-adjusting its step size to accommodate a given accuracy requirement, thus preventing unstable error propagation and enabling the revelation of the detailed behavior of the solution at those interesting parts where there is a great deal of structure. However, the predictor-corrector method, though powerful, is not self-starting; hence, the Runge-Kutta method is used as a starter to generate the first four points. With these methods, program SATEDYN (listed in this chapter) is capable of generating digital simulations of the time-dependent dynamical behavior of the satellite system, depending on the experimental conditions. Within this program, the coupled second order nonlinear differential equations are transformed to a set of coupled first order differential equations. The first derivatives of the seven variables used are defined as additional variables, resulting in a set of fourteen (14) coupled first order nonlinear differential equations in fourteen (14) variables. The details of the input formats are listed in Table 6.

Another method of simulation is used in the program SYNHARM, using a frequency approach. It lacks the full fledge capability of program SATEDYN, but has the advantage of simplicity. If the normal frequency of a system are well known, then for harmonic oscillations near equilibrium, the amplitude of each generalized coordinate can be equated to a Fourier series in terms of the normal frequencies. A complete set of such Fourier series equations then forms the harmonic equations of motion of the system. Time-dependent solutions can then be generated to match a given complete set of initial conditions. The details of the input formats for program SYNHARM are listed in Table 7.

Table 6. Inputs for Program SATEDYN

Variable Name	Data Card Number	Data Card Column	Format	Variable Description
RB(I),[I=1, ..4]	1	1-40	4F10.2	Initial boom lengths (ft.)
TAU	2	1-10	F10.2	Deployment/Retraction period (Sec.)
IPLOY (I) [I=1,...4]	2	11-18	4I2	Ith boom undergoes deployment if IPLOY (I) = 1, retraction if IPLOY (I) = -1, and no deployment/retraction if IPLOY (I) =0
Y (2*I + 1)	3	1-40	4F10.2	Initial boom deflections (rad.)
IRPM	4	1	I1	IRPM = 0: if initial RPM is given = 1: if final RPM is given = 2: if the final values of a previous run are used as initial values for this run
RPM	4	2-11	F10.4	Revolution per minute of hub. (Range: 3 to 9 RPM)
MPLOT	4	12-13	I2	MPLOT = 0: if pen plots of power spectra is not desired 1: if pen plots of power spectra is desired
MHIS	4	14-18	I2	MHIS = 0: if histogram of power spectra is not desired 1: if histogram of power spectra is desired
PRMT (2)	5	1-10	F10.2	Total time of simulation (Sec.), (Note: not to be confused with computer time). Usual range is 600 to 4200 sec. If this range limit is exceeded, dimensions in the program would have to be adjusted accordingly.
Y(I)	6-20	2-21	F20.10	Initial values of position and velocities of seven harmonic variables
[I = 1, ...14]		for each card		Note: these cards (#6-20) are needed only if (IRPM=2) the final values of a previous run are used as initial values for this run. If these cards are used, they overwrite the four data Y(2*I + 1) read earlier on card 3.

Table 7. Inputs for Program SYNHARM

Variable Name	Data Card Number	Data Card Column	Format	Variable Description
NO	1	1-2	I2	Total number of data sets
OMEGAO	2	1-10	F10.5	Hub Spin (rad/sec)
R	2	11-20	F10.5	Wire boom length (ft.) (Equal boom length assumed for this program)
IO	2	21-22	I2	ID = 0: for simulation of harmonic oscillations out of spin plane, 1: for simulation of harmonic oscillations in spin plane
P(I), (I=1, ...7)	3	1-80	7F10.5	Initial values of positions of seven variables for harmonic oscillations. (Angular variables in radians, and translational variables in ft.)
V(I), (I=1, ...7)	4	1-80	7F10.5	Initial values of velocities of seven variables for harmonic oscillations (Angular variables in rad/sec, and translational variables in ft/sec)
IFFT	5	1-2	I2	IFFT =0: fast Fourier analysis not desired, 1: fast Fourier analysis desired
IM	5	3-4	I2	IM=0: if max. and min. of ordinates in plots are ± 0.15 1: automatic scaling of ordinates in plots
ISPEC	5	5-6	I2	ISPEC =0: if histogram of power spectra is desired 1: if histogram of power spectra is not desired

Program SATEDYN listing follows.

```
      PROGRAM SATEDYN( INPUT, OUTPUT, TAPE5=INPUT, TAPE6=OUTPUT, PUNCH ) SATEDYN   2
C                                                                       SATEDYN   3
C     PROGRAM TO SIMULATE SATELLITE WIRE-BOOM DYNAMICS IN SPIN PLANE,   SATEDYN   4
C     INCLUDING DEPLOYMENT/RETRACTION, INITIAL BOOM DEFLECTIONS, AND    SATEDYN   5
C     ANGULAR VELOCITIES.                                               SATEDYN   6
C                                                                       SATEDYN   7
C     PREPARED BY LOGICON, INC., DEC. 1974,                             SATEDYN   8
C     FOR THE ANALYSIS AND SIMULATION BRANCH, AIR FORCE CAMBRIDGE       SATEDYN   9
C     RESEARCH LABORATORIES, L. G. HANSCOM FIELD, BEDFORD, MASS.        SATEDYN  10
C                                                                       SATEDYN  11
C     THIS PROGRAM CALLS THE FOLLOWING IBM SCIENTIFIC SUBROUTINES (5TH  SATEDYN  12
C     EDITION, 1970) :-                                                 SATEDYN  13
C     HPCG                                                              SATEDYN  14
C     RHARM                                                             SATEDYN  15
C     HARM                                                              SATEDYN  16
C     SIMQ                                                              SATEDYN  17
C                                                                       SATEDYN  18
      EXTERNAL FCT, OUTP, DAMPER                                        SATEDYN  19
      DIMENSION PRNT(10), Y(50), DERY(14), AUX(16,14)                   SATEDYN  20
      DIMENSION RB(4), RFB(4), RDOTB(4), A(4), B(4)                     SATEDYN  21
      DIMENSION R(4), RF(4), RDOT(4), PROGID(3), IPLOY(4)               SATEDYN  22
      COMMON/SATCOM/ R, RO, RDOT, RHO, AM, AIO, AITO, DA4, DAMP, ALPHMX, SATEDYN  23
     1              AMO, DELTA(4), AKPHI, S, OMEGAO, TAJ, RF, ISET      SATEDYN  24
      COMMON /DDD/ KAMP(4), KEY(4)                                      SATEDYN  25
      COMMON/FFT/ NPLOT, NHIS                                           SATEDYN  26
      EQUIVALENCE (OMEGAO, OMEGA0)                                      SATEDYN  27
C                                                                       SATEDYN  28
C                                                                       SATEDYN  29
C     Y(2*I-1) AND Y(2*I) ARE ANGULAR DEFLECTIONS AND VELOCITIES OF ITH SATEDYN  30
C     BOOM (I = 1, ... 4).                                              SATEDYN  31
C     Y(11), Y(12), Y(13), Y(14) , ARE X, XDOT, Y, YDOT, RESPECTIVELY.  SATEDYN  32
C     Y(15) TO Y(18) ARE PHIMINS                                        SATEDYN  33
C     Y(19) TO Y(22) ARE TENSIONS                                       SATEDYN  34
C     Y(23), TO Y(34) ARE OLD VALUES OF Y(3) TO Y(14)                   SATEDYN  35
C     Y(35), ... Y(38) ARE ROOTS                                        SATEDYN  36
C     Y(39) = TAU                                                       SATEDYN  37
C     Y(45) = 399 IF SYSTEM IS UNBALANCED WITH DEPLOYMENT/RETRACTION.   SATEDYN  38
C     Y(49) = FINAL RADIAN PER SEC. OF HUB                              SATEDYN  39
C     Y(50) = PI                                                        SATEDYN  40
C                                                                       SATEDYN  41
C     INPUT DATA                                                        SATEDYN  42
C                                                                       SATEDYN  43
      PI = 4.*ATAN(1.)                                                  SATEDYN  44
      NDIM = 14                                                         SATEDYN  45
      ISET = 0                                                          SATEDYN  46
      PROGID(1) = 10HLOGICON                                            SATEDYN  47
      PROGID(2) = 10HSATEDYN                                            SATEDYN  48
      PROGID(3) = 10H     LAI                                           SATEDYN  49
      AMB = 2.0                                                         SATEDYN  50
C     AMOB = 430.9 LB., GIVEN.                                          SATEDYN  51
      AMOB = 430.9                                                      SATEDYN  52
      ROB=  2.57                                                        SATEDYN  53
      RHO=  0.0945                                                      SATEDYN  54
      RHO= 0.0945*30.48/453.5924                                        SATEDYN  55
C     AIOB = 115.7 SLUGS (IN-PLANE), GIVEN.                             SATEDYN  56
      AIOB = 115.7*32.0                                                 SATEDYN  57
      AKPHI = 10000.0                                                   SATEDYN  58
```

```
      SB = 1.0                                                          SATEDYN   59
      S  =  6.0*980.0*9.5/8.0*180.0/(3.1416)                            SATEDYN   60
C                                                                       SATEDYN   61
C     ASSUMED COULOMB DAMPER CHARACTERISTICS :                          SATEDYN   62
C     LIMITING FRICTION = 4.0 GRAMS                                     SATEDYN   63
C     ALPHA MAX = .0648 RADIANS                                         SATEDYN   64
C                                                                       SATEDYN   65
      DAMP = 4.0*980.0                                                  SATEDYN   66
      DAM = DAMP*9.5                                                    SATEDYN   67
      ALPHMX = .0648                                                    SATEDYN   68
C                                                                       SATEDYN   69
      GM = 453.5924                                                     SATEDYN   70
      CM = 30.48                                                        SATEDYN   71
   28 READ(5,29) LSET                                                   SATEDYN   72
   29 FORMAT(I2)                                                        SATEDYN   73
      READ(5,55) (RB(I),  I=1,4)                                        SATEDYN   74
      READ(5,27)TAU, (IPLOY(I), I=1,4 )                                 SATEDYN   75
   27 FORMAT( F10.2, 4I2)                                               SATEDYN   76
      DO 727 I = 1, 4                                                   SATEDYN   77
      RDOTB(I) = FLOAT( IPLOY(I) )*0.05                                 SATEDYN   78
      RDOT(I) = CM*RDOTB(I)                                             SATEDYN   79
  727 RFB(I) = RDOTB(I)*TAU + RB(I)                                     SATEDYN   80
   55 FORMAT(4F10.2)                                                    SATEDYN   81
      WRITE(6,11)                                                       SATEDYN   82
   11 FORMAT(1H1, 1X, *INPUT CARDS 1-*, //)                             SATEDYN   83
      WRITE(6,29) LSET                                                  SATEDYN   84
      WRITE(6,55) (RB(I), I=1, 4)                                       SATEDYN   85
      WRITE(6,27) TAU, (IPLOY(I), I=1, 4)                               SATEDYN   86
      RHOB = RHO*CM/GM                                                  SATEDYN   87
      CALL PLTID3( PROGID, 300.0, 12.0, 0.90)                           SATEDYN   88
C                                                                       SATEDYN   89
C     SUBSCRIPTS 1, 2, 3, 4, ARE THE BOOM NUMBERS                       SATEDYN   90
C                                                                       SATEDYN   91
      DO 22 I=1,4                                                       SATEDYN   92
      DELTA(I) = (FLOAT(I)-1.0)*0.5*PI                                  SATEDYN   93
      R(I) = CM*RB(I)                                                   SATEDYN   94
   22 RF(I) = CM*RFB(I)                                                 SATEDYN   95
      RO = CM*ROB                                                       SATEDYN   96
      AM = GM*AMB                                                       SATEDYN   97
      AMO= GM*AMOB                                                      SATEDYN   98
      AKPHIB = AKPHI/(GM*CM**2)                                         SATEDYN   99
      AIO = AIOB*CM**2*GM                                               SATEDYN  100
      AITF = AIO                                                        SATEDYN  101
      AITO = AIO                                                        SATEDYN  102
      DO 33 I = 1, 4                                                    SATEDYN  103
      A(I) = AM*R(I)**2 + RHO*R(I)**3/3.0                               SATEDYN  104
      B(I) = AM*R(I)     + RHO*R(I)**2/2.0                              SATEDYN  105
   33 AITO = AITO + A(I) + (AM+RHO*R(I))*RO**2 + 2.0*B(I)*RO            SATEDYN  106
      DO 20 I = 1, 4                                                    SATEDYN  107
      A(I) = AM*RF(I)**2 + RHO*RF(I)**3/3.0                             SATEDYN  108
      B(I) = AM*RF(I) + RHO*RF(I)**2/2.0                                SATEDYN  109
   20 AITF = AITF + A(I) + (AM+RHO*RF(I))*RO**2 + 2.0*B(I)*RO           SATEDYN  110
      AITOB = AITO/(GM*CM**2)                                           SATEDYN  111
   10 DO 2 I = 1, 4                                                     SATEDYN  112
      KEY(I) = 0                                                        SATEDYN  113
    2 KAMP(I) = 0                                                       SATEDYN  114
      ISET = ISET + 1                                                   SATEDYN  115
```

```
         DO 65 I = 1, 50                                                  SATEDYN   116
      65 Y(I) =0.0                                                        SATEDYN   117
         READ(5,55) (Y(2*I+1), I = 1, 4  )                                SATEDYN   118
         WRITE(6,55) (Y(2*I+1), I=1, 4)                                   SATEDYN   119
C                                                                         SATEDYN   120
C        OPTION -- IRPM = 0 : IF INITIAL RPM IS GIVEN.                    SATEDYN   121
C                       = 1 : IF FINAL RPM IS GIVEN.                      SATEDYN   122
C                       = 2 : IF THE FINAL VALUES OF THE PREVIOUS RUN ARE SATEDYN   123
C                             USED AS INITIAL VALUES FOR THIS RUN.        SATE'     124
C                                                                         SATE      125
C        OPTION : --- MPLOT = 1 FOR POWER SPECTRA PEN PLOTS               SATE.     126
C                     MHIS = 1 FOR POWER SPECTRA HISTOGRAMS               SATED.    127
C                                                                         SATEDYN   128
         READ(5,15) IRPM, RPM, MPLOT, MHIS                                SATEDYN   129
         WRITE(6, 15) IRPM, RPM, MPLOT, MHIS                              SATEDYN   130
      15 FORMAT(1X, I1, F10.4, 2I2)                                       SATEDYN   131
         READ(5,753) PRMT(2)                                              SATEDYN   132
         WRITE(6,753) PRMT(2)                                             SATEDYN   133
     753 FORMAT(F10.2)                                                    SATEDYN   134
         IF (IRPM-1) 123, 124, 125                                        SATEDYN   135
     123 OMEGAO = 2.0*PI*RPM/60.0                                         SATEDYN   136
         CALL SYMBOL( 1.16, 9.5, 0.11, 7HINITIAL, 0.0, 7 )                SATEDYN   137
         GO TO 128                                                        SATEDYN   138
     124 OMEGAO = AITF*RPM/AITO*2.0*PI/60.0                               SATEDYN   139
         CALL SYMBOL( 1.16, 9.5, 0.11, 5HFINAL, 0.0, 5 )                  SATEDYN   140
     128 Y(2) = OMEGAO                                                    SATEDYN   141
         CALL NUMBER( 0.94, 9.5, 0.11, RPM, 0.0, -1 )                     SATEDYN   142
         GO TO 125                                                        SATEDYN   143
     125 DO 21 I = 1, 14                                                  SATEDYN   144
         READ(5,502) Y(I)                                                 SATEDYN   145
      21 WRITE(6,502) Y(I)                                                SATEDYN   146
         OMEGAO = Y(2)                                                    SATEDYN   147
         RPM = 60.0*OMEGAO/(2.0*PI)                                       SATEDYN   148
         CALL SYMBOL( 1.05, 9.5, 0.11, 8HAS IT IS, 0.0, 8 )               SATEDYN   149
     126 DSUM = 0.0                                                       SATEDYN   150
         CALL SYMBOL( 0.5, 9.5, 0.11, 4HRPM=, 0.0, 4 )                    SATEDYN   151
         CALL SYMBOL( 0.5, 9.0, 0.11, 17H  ,  ,  ,   FT.--, 0.0, 17 )     SATEDYN   152
         CALL SYMBOL( 0.5, 8.5, 0.11, 15H  ,  ,  ,   FT., 0.0, 15 )       SATEDYN   153
         AX = 0.5                                                         SATEDYN   154
         DO 86 I = 1, 4                                                   SATEDYN   155
         CALL NUMBER( AX, 9.0, 0.11, RB(I), 0.0, -1 )                     SATEDYN   156
         CALL NUMBER( AX, 8.5, 0.11, RFB(I),0.0, -1 )                     SATEDYN   157
         Y(34+I) = RROT(I)                                                SATEDYN   158
      86 AX = AX + 0.33                                                   SATEDYN   159
         CALL PLOT( 4.0, 0.0, -3 )                                        SATEDYN   160
         WRITE(6,834)                                                     SATEDYN   161
     834 FORMAT(//////)                                                   SATEDYN   162
C        DO 501 I = 1, 50                                                 SATEDYN   163
C    501 WRITE(5,502) Y(I)                                                SATEDYN   164
     502 FORMAT(1X, F20.10)                                               SATEDYN   165
         Y(39) = TAU                                                      SATEDYN   166
         Y(50) = PI                                                       SATEDYN   167
         Y(49) = RPM*2.0*PI/60.                                           SATEDYN   168
C                                                                         SATEDYN   169
C        PRMT(7) OPTIONS :-                                               SATEDYN   170
C        OPTION -- 0 : NO PUNCHED CARD OUTPUT.                            SATEDYN   171
C                   1 : PUNCHED CARD OUTPUT OF FINAL VALUES OF THIS RUN FOR SATEDYN 172
```

PROGRAM SATEDYN   74/74   OPT=1   FTN 4.1+P378   12/20/74 16.32.40.   PAGE   4

```
C                   INITIAL VALUES OF NEXT RUN.
      PRMT(1) = 0.0
      PRMT(3) = 1.0
      PRMT(4) = 0.00001
      PRMT(6) = RPM
      PRMT(8) = IHARM
      PRMT(9) = ISET
      PRMT(10) = -2.0
      WRITE(6,73)
      WRITE(6,74)
      WRITE(6,7) ISET
      WRITE(6,4)  (RB(I),  R(I),  I, I = 1, 4)
      WRITE(6,38) (RFB(I), RF(I), I, I = 1, 4)
      WRITE(6,44) (RDOTB(I), RDOT(I), I, I=1, 4)
      WRITE(6,5) AIOB, AIO
      WRITE(6,55) AITOB, AITO
      WRITE(6,5) AMB, AM,  AMOB, AMO,  ROB, RO,  RHOB, RHO
      WRITE(6,77) TAU
      WRITE(6,5) OMEGAO, RPM, AKPHIB, AKPHI, SB, S
      WRITE(6,39)
      WRITE(6,1)
      WRITE(6,79)
      IF (RDOT(1).EQ.RDOT(3). AND. RDOT(2).EQ.RDOT(4)) GO TO 60
      Y(45) = 999.
   60 DO 14 I = 1, NDIM
   14 DERY(I) = 1.0/FLOAT(NDIM)
C
C     DETERMINE INITIAL TRANSLATIONAL VELOCITIES
C
      CALL BAL( Y, R, DELTA, RDOT, AM, AMO, RO, RHO )
C
C     SOLVE COUPLED NONLINEAR DIFFERENTIAL EQUATIONS
C
      CALL HPCG( PRMT, Y, DERY, NDIM, IHLF, FCT, OUTP, DAMPER, AUX )
      IF (Y(45).NE.999) GO TO 212
      Y(45) = 0.0
      DO 62 I=1, 4
      RDOT(I) = 0.0
      R(I) = RF(I)
   62 Y(34+I) = 0.0
  212 CONTINUE
      IF (PRMT(5).EQ.1.0) GO TO 60
      WRITE (6,9) IHLF, (I, PRMT(I), I = 1, 5)
C     DO 303 J = 1, 16
C     WRITE(6,3) (J, K, AUX(J,K), K = 1,4)
C 303 CONTINUE
C
C     REPEAT CALCULATIONS FOR A DIFFERENT SET OF INPUT DATA
C
      IF (ISET.LT..SET) GO TO 28
   72 CALL ENDPLT
C
C     FORMATS ARE GIVEN AS FOLLOWS :-
C
   73 FORMAT(///////, 1X, *DAMPER ACTION SYMBOLS --- *, /,
     1T29,*0 = NO DAMPING, ANGLE TOO SMALL.*, / ,
     2T29,*1 = DAMPER IN ACTION.*,/,T29, *2 = NO DAMPING, ANGLE TOO LARG
```

```
      3E.*,//)                                                          SATEDYN  230
    1 FORMAT(////,1X, 25HOUTPUT DATA : ..........  ,//,T4, *TIME*,      SATEDYN  231
     1T12, *PHI 1*, T21, *PHI 2*, T30, *PHI 3*, T39, *PHI 4*, T48, *HUB SATEDYN  232
     2VEL*, T55, *X*, T65, *X-VEL*, T76, *Y*, T84, *Y-VEL*, T92, *DAMPER SATEDYN  233
     3S*, T102, *TENSIONS*, T119, *NO.*)                                SATEDYN  234
    3 FORMAT( 1X, 4HAUX(, I2, 1H,, I1, 4H) = , F10.3)                   SATEDYN  235
    4 FORMAT(1X, *INITIAL WIRE LENGTH*, T27, *=*, F13.5, T42, *FT.*, T56 SATEDYN  236
     1, *=*, F17.5, T75, *CM.*, T90, 1H(, I2, 2H ))                     SATEDYN  237
    5 FORMAT (1X, *INITIAL ANGULAR VELOCITY*, T27, *=*, F13.5, T42,     SATEDYN  238
     1*RAD/SEC*, T56, *( FINAL ANG. VEL.=*, F8.5, * RPH )*, /,          SATEDYN  239
     2        1X, *KPHI*, T27, *=*, F13.5, T42, *LB.FT.2/SEC.RAD.*, T60, SATEDYN  240
     3*=*, F13.5, T75, *GM.CM.2/SEC.RAD.*, /,                           SATEDYN  241
     41X,*S*, T27, *=*, F13.5, T42, *LB.FT.2/RAD.SEC.2*,      *=*,      SATEDYN  242
     5F13.1, T75, *GM.CM.2/RAD.SEC.2*,/)                                SATEDYN  243
   44 FORMAT(1X, *RATE OF DEPLOYMENT*, T27, *=*, F13.5, T     *FT./SEC.*, SATEDYN  244
     1T56, *=*, F17.5, T75, *CM./SEC.*, T90, 1H(, I2, 2H ),             SATEDYN  245
    6 FORMAT(                                                           SATEDYN  246
     11X, *MOMENT OF INERTIA OF HUB*, T27, *=*, F13.5, T42, *LB.FT.2*,  SATEDYN  247
     2T56, *=*, F17.1, T75, *GM.CM.2.*)                                 SATEDYN  248
    7 FORMAT(1H1, 37HINPUT DATA FOR S3-2 SATELLITE PROBLEM,3X, 9H....... SATEDYN  249
     1..,T100, *........DATA SET *, I2, ///)                            SATEDYN  250
    8 FORMAT(1X, *TIP MASS*, T27, *=*, F13.5, T42, *LB.*, T56, *=*, F17. SATEDYN  251
     15, T75, *GM.*, /,                                                 SATEDYN  252
     21X, *MASS OF HUB*, T27, *=*, F13.5, T42, *LB.*, T56, *=*, F17.5,  SATEDYN  253
     3T75, *GM.*, /,                                                    SATEDYN  254
     41X, *RADIUS OF HUB*, T27, *=*, F13.5, T42, *FT.*, T56, *=*, F17.5, SATEDYN  255
     5T75, *CM.*, /,                                                    SATEDYN  256
     61X, *MASS DENSITY OF BOOM*, T27, *=*, F13.5, T42, *LB./FT.*, T56, SATEDYN  257
     7*=*, F17.5, T75, *GM./CM.* )                                      SATEDYN  258
    9 FORMAT(1H1,1X,8HIHLF    =,I5,/,5(1X,5HPRMT(,I1,4H) = ,F10.5,/))   SATEDYN  259
   66 FORMAT(1X,*TOTAL MOMENT OF INERTIA*, T27, *=*, F13.4, T42, *LB.FT. SATEDYN  260
     12*, T56, *=*, F17.1, T75, *GM.CM.2.* )                            SATEDYN  261
   74 FORMAT(2X, *TENSION SIGN SYMBOLS ---- *, /,                       SATEDYN  262
     1T29, *RATIO OF TENSION(PHI,X,Y) TO TENSION(PHI=0,X=0,Y=0)*)       SATEDYN  263
   77 FORMAT(1X,*PERIOD OF DEPLOYMENT*,T27,*=*,F13.5,T42,*SEC.*)        SATEDYN  264
C  78 FORMAT(T4,*SEC.*,T12,*RAD.*,T19,*RAD/SEC*,T29,*RAD.*,T35,*RAD/SEC* SATEDYN  265
C    1,T45,*RAD.*,T51,*RAD/SEC*,T61,*RAD.*,T67,*RAD/SEC*,T76,*RA/SEC*,T8 SATEDYN  266
C    24, *CM.*,T94, *CM.*, /)                                           SATEDYN  267
   78 FORMAT(T4, *SEC.*, T13, *RAD.*, T22, *RAD.*, T31, *RAD.*, T40,    SATEDYN  268
     1*RAD.*, T47, *RAD/SEC*, T58, *CM.*, T65, *CM/SEC*, T76, *CM.*, T83 SATEDYN  269
     2,*CM/SEC*, T92, *ON-OFF*, T102, *RATIOS*,/)                       SATEDYN  270
   88 FORMAT(1X, *FINAL WIRE LENGTH*, T27, *=*, F13.5, T42, *FT.*,T56,  SATEDYN  271
     1*=*, F17.5, T75, *CM.*, T90, 1H(, I2, 2H ))                       SATEDYN  272
   99 FORMAT(1X, 22H(CONVERSION FACTORS : , 19H1 LB.=453.5924 GM.,,     SATEDYN  273
     117H 1 FT.=30.48 CM.),/)                                           SATEDYN  274
      STOP                                                              SATEDYN  275
      END                                                               SATEDYN  276
```

```
      A(5,7) = A(5,7)+BT(I)*COS(YDDD) + DT(I)*RO *COS(DELLL(I))          FCT   59
      A(6,6) = A(6,6) + DT(I)                                           FCT   60
      Z1 = BT(I)*(RDOT(I)-RO*SIN(Y(J))*Y(2*I+2))                        FCT   61
      Z1 = 2.0*(Y(2)+Y(J+1))*( DT(I)*RDOT(I)+RO*COS(Y(J))+Z1 )          FCT   62
      Z2 = BT(I)*RO*SIN(Y(J))*Y(2*I+2)**2                               FCT   63
      Z3 = RHO*(Y(2)*RO**2*RDOT(I) + RO*RDOT(I)**2*SIN(Y(J)))           FCT   64
      Z4= -RHO*RO*RDOT(I)*(Y(14)*COS(DELLL(I)) - Y(12)*SIN(DELLL(I)))   FCT   65
      B(5) = B(5)-Z1-Z2-Z3+Z4                                           FCT   66
      Z1 =-BT(I)*(2.0*RDOT(I)*(Y(2*I+2)+Y(2)) + RO*SIN(Y(J))*Y(2)**2 )  FCT   67
      Z2 = AKPHT*Y(2*I+2) + S*Y(J)                                      FCT   68
      B(I) = Z1-Z2                                                      FCT   69
      B(6) = B(6) - RHO*RDOT(I)*(Y(12)+RDOT(I)*COS(YDDD)) +DT(I)*RDOT(I) FCT   70
     1*2.0*(Y(2)+Y(K))*SIN(YDDD)+BT(I)*(Y(2)+Y(K))**2*COS(YDDD)          FCT   71
     2+RHO*RDOT(I)*RO*Y(2)*SIN(DELLL(I))                                FCT   72
     3+DT(I)*RO*COS(Y(1)+DELTA(I))*Y(2)**2                              FCT   73
      B(7) = B(7) - RHO*RDOT(I)*(Y(14)+RO*Y(2)+RDOT(I)*SIN(YDDD))       FCT   74
     1-DT(I)*RDOT(I)*2.0*(Y(2)+Y(K))*COS(YDDD)+BT(I)*(Y(2)+Y(K))**2*SIN( FCT   75
     2YDDD)+DT(I)*RO*SIN(DELLL(I))*Y(2)**2                              FCT   76
C                                                                       FCT   77
C     TEST WHETHER THERE IS DAMPING BY COULOMB DAMPER                   FCT   78
C                                                                       FCT   79
   33 CALL DAMPER(X, Y, DAM, I, B, PHC)                                 FCT   80
      A(7,7) = A(6,6)                                                   FCT   81
      A(5,5) = A(5,6)                                                   FCT   82
      A(7,5) = A(5,7)                                                   FCT   83
      A(7,6) = A(6,7)                                                   FCT   84
C                                                                       FCT   85
C     SOLVE SIMULTANEOUS EQUATION AX=B                                  FCT   86
C                                                                       FCT   87
      CALL SIM2(A,B,7,KS)                                               FCT   88
      IF (KS.NE.0) WRITE(6,34) X, (B(I), I=1,7)                         FCT   89
   34 FORMAT(1X, F10.2, 7F12.4, T118,*XX*)                              FCT   90
      DERY(2) = B(5)                                                    FCT   91
      DERY(12)= B(6)                                                    FCT   92
      DERY(14)= B(7)                                                    FCT   93
      DO 38 I = 1,4                                                     FCT   94
   38 DERY(2*I+2) = B(I)                                                FCT   95
      DO 73 I = 1, 13, 2                                                FCT   96
   73 DERY(I) = Y(I+1)                                                  FCT   97
      DO 77 I = 1, 8                                                    FCT   98
   77 Y(22+I) = Y(2+I)                                                  FCT   99
      RETURN                                                            FCT  100
      END                                                               FCT  101
```

```
      SUBROUTINE OUTP( X, Y, DERY, IHLF, NDIM, PRMT)                    OUTP      2
      DIMENSION Y(1), DERY(1), PRMT(1)                                  OUTP      3
      DIMENSION AA(5), BB(5), TTEN(5)                                   OUTP      4
      DIMENSION FREQ(552)                                               OUTP      5
      DIMENSION XX(2100), YY(2100), WW(2100), PP(2100), QQ(2100),       OUTP      6
     1ZZ(6300)                                                          OUTP      7
      COMMON /DDD/ KAMP(4), KEY(4)                                      OUTP      8
C                                                                       OUTP      9
C     PRMT(10) = XOLD                                                   OUTP     10
C     PN = NO. OF POINTS TO BE PLOTTED IN X AND Y PLOTS                 OUTP     11
C                                                                       OUTP     12
      IF (X.GT.1.9) GO TO 1                                             OUTP     13
      PN = PRMT(2)                                                      OUTP     14
      IF (PRMT(2).GT.2100.0) DT = 2.0                                   OUTP     15
      DT = 1.0                                                          OUTP     16
      NP = INT(PN/DT+0.0000001) + 1                                     OUTP     17
      DTT = DT-0.0000001                                                OUTP     18
      IF (PN.EQ.600.) PN = 350.                                         OUTP     19
      YMIN = 1.E+30                                                     OUTP     20
      YMAX =-1.E+30                                                     OUTP     21
    1 DX = X - PRMT(10)                                                 OUTP     22
      DTPLOT = PRMT(2) - X                                              OUTP     23
      IF (DTPLOT.LE.1.1) DTT = 0.99                                     OUTP     24
      IF (DX.LE.DTT) GO TO 8                                            OUTP     25
      IF (Y(45).NE.999) GO TO 100                                       OUTP     26
      IF (X.GT.Y(39). AND. X.LE.(Y(39)+DT)) PRMT(5) = 1.0               OUTP     27
      IF (X.GT.Y(39). AND. X.LE.(Y(39)+DT)) PRMT(1) = X                 OUTP     28
  100 CONTINUE                                                          OUTP     29
      II = II+1                                                         OUTP     30
      MODII = MOD(II,10)                                                OUTP     31
      IF (MODII.EQ.1) WRITE (6,3)                                       OUTP     32
    3 FORMAT(1X, / )                                                    OUTP     33
      DO 33 I=1,4                                                       OUTP     34
      TTEN(I) = Y(18+I)/Y(40+I)                                         OUTP     35
   33 CONTINUE                                                          OUTP     36
C     WRITE(6,5) X, Y(3),Y(4),Y(5),Y(6),Y(7),Y(8),Y(9),Y(10),Y(2),Y(11),OUTP     37
C    1Y(13), (KAMP(III), III=1,4), (TTEN(III), III=1,4)                 OUTP     38
C   6 FORMAT(1X, F5.2, 8F8.4, F7.4, 2F9.4, 1X, 4I2, 1X, 4F4.1)          OUTP     39
C                                                                       OUTP     40
C     OUTPUT FORMAT FOR SIMULATION TIME SERIES                          OUTP     41
C     OPTION -- IJ = 0 : OUTPUT DATA INCLUDE TENSION RATIOS, OMIT THETA.OUTP     42
C                    1 : OUTPUT DATA INCLUDE THETA, OMIT TENSION RATIOS.OUTP     43
C                                                                       OUTP     44
      IF (IJ.EQ.0) GO TO 99                                             OUTP     45
      TWOPI = 2.*4.*ATAN(1.)                                            OUTP     46
      THETA = AMOD( Y(1), TWOPI)*360./TWOPI                             OUTP     47
      WRITE(6,19)X, Y(3), Y(5), Y(7), Y(9), Y(2), Y(11), Y(12), Y(13),  OUTP     48
     1Y(14), THETA, II                                                  OUTP     49
   19 FORMAT(1X, F7.2, 4F9.4, F10.6, F8.4, 3F9.4, 2X, F10.4, I5)        OUTP     50
      GO TO 5                                                           OUTP     51
   99 CONTINUE                                                          OUTP     52
C     WRITE(6,9) X, Y(3), Y(5), Y(7), Y(9), Y(2), Y(11), Y(13), Y(19),  OUTP     53
C    1Y(20), Y(21), Y(22)                                               OUTP     54
C   9 FORMAT(1X, 8F9.4, 4F11.2)                                         OUTP     55
      WRITE(6,9) X, Y(3), Y(5), Y(7), Y(9), Y(2), Y(11), Y(12), Y(13),  OUTP     56
     1Y(14), (KAMP(III), III=1,4), (TTEN(III), III=1,4), II             OUTP     57
    9 FORMAT(1X, F7.2, 4F9.4, F10.6, F8.4, 3F9.4, 2X, 4I2, 2X, 4F4.1, I5OUTP     58
```

```
      1)
    5 YY(II) = Y(3)
      WW(II) = Y(5)
      PP(II) = Y(7)
      QQ(II) = Y(9)
      XX(II) = X
      IF (X.GE.PN ) GO TO 44
      ZZ(II) = Y(2)
      IF (YMIN.GT.ZZ(II,) YMIN=ZZ(II)
      IF (YMAX.LT.ZZ(II)) YMAX=ZZ(II)
      ZZ( NP+II) = Y(11)
      ZZ(NP*2+II) = Y(13)
      PN1 = PN - 1.0
      IF (X.LT.PN1) GO TO 44
      IPLOT = 1
      XMIN = 0.0
      DX = 100.0
      IF (PN.EQ.350.) DX = 50.
      AA(1) = 0.0
      AA(2) = PN
      AA(3) = PN
      AA(4) =0.0
  355 CONTINUE
      PLOTI = IPLOT
C
C     IPLOT IS PLOT NUMBER #.
C     SCALING Y-AXIS OF PLOTS #1, 2, 3 (X-AXIS IS TIME, IN SEC.)
C
      T = YMAX-YMIN
      YMIN = YMIN-0.1*T
      IF (T.LT.0.001) YMIN = AINT((YMIN-0.0005)*2000.0)/2000.
      IF (T.LT.0.01. AND. T.GT.0.001) YMIN=AINT((YMIN-0.005)*200.)/200.
      IF (T.GE.0.01) YMIN = AINT((YMIN-0.05)*20.)/20.0
      YMAX = YMAX + 0.1*T
      IF (T.LT.0.01) YMAX = AINT((YMAX+ 0.005)*200.0)/200.0
      IF (T.GE.0.01) YMAX = AINT((YMAX+0.05)*20.)/20.0
      IF (IPLOT.EQ.1) GO TO 330
      YMM = YMIN/(YMAX+1.E-30)
      IF (YMM.GT.0.0) GO TO 330
      T = -YMIN
      IF (YMM.GE.-1.0) T=YMAX
      IF (T.GT.25.0) T = AINT(T/25.0)*25.0+25.0
      IF (T.GT.5.0. AND. T.LT.25.0) T=AINT(T/5.0)*5.0+5.0
      IF (T.LT.5.0) T=5.0
      YMAX = T
      YMIN =-T
  330 DY = (YMAX-YMIN)/10.0
C     WRITE(6,331) YMIN, YMAX, DY, IPLOT
  331 FORMAT(//,1X,*YMIN=*,F10.5,*YMAX=*,F10.5,10X,*DY=*,F10.5,*IPLOT=*,
     1I4,//)
      BB(1) = YMAX
      BB(2) = YMAX
      BB(3) = YMIN
      BB(4) = YMIN
      CALL LINE( AA, BB, 4, 1, 0, 3, XMIN, DX, YMIN, DY, 0.08 )
      CALL SYMBOL( 2.5, 0.28,0.25, 10HTIME(SEC.), 0., 10 )
      DO 12 I = 1, 5
```

```
      T = FLOAT(2*I) - 1.0                                                OUTP   116
   12 CALL SYMBOL( 0., T, 0.1, 13, 90.0, -1 )                             OUTP   117
      IN = PN/DX                                                          OUTP   118
      DO 10 I = 1, IN                                                     OUTP   119
      T = FLOAT(I)                                                        OUTP   120
   10 CALL SYMBOL( T, 0., 0.1, 13, 0.0, -1 )                              OUTP   121
      IN = INT( PN/DX*0.5 ) + 1                                           OUTP   122
      DO 11 I = 1, IN                                                     OUTP   123
      T = FLOAT(2*I-2) - 0.1                                              OUTP   124
      FPN = 2.*DX*(FLOAT(I)-1.0)                                          OUTP   125
   11 CALL NUMBER( T, 0.05, 0.19, FPN, 0.0, -1 )                          OUTP   126
      CALL AXIS(0., 0., 1H , 1, 10., 90., YMIN, DY, 5.)                   OUTP   127
      XMAX = PN/DX                                                        OUTP   128
      CALL AXIS(XMAX,0., 1H , 1, 10., 90., YMIN, DY, 5.)                  OUTP   129
      CALL LINE (XX, ZZ, II, 1, 0, 3, XMIN, DX, YMIN, DY, 0.08)           OUTP   130
      IF (IPLOT.GT.1) GO TO 41                                            OUTP   131
C                                                                         OUTP   132
C     PLOT HUB SPIN RATE                                                  OUTP   133
C                                                                         OUTP   134
      CALL SYMBOL( -0.82,  3.5, 0.25, 11H(RAD./SEC.), 90., 11)            OUTP   135
      CALL SYMBOL(1.2, 10.2, .25, 23HANGULAR VELOCITY OF HUB, 0.0, 23)    OUTP   136
   41 CONTINUE                                                            OUTP   137
      T = PN/DX + 3.0                                                     OUTP   138
      CALL PLOT( T, 0.0, -3 )                                             OUTP   139
      IPLOT = IPLOT + 1                                                   OUTP   140
      YMIN = 0.0                                                          OUTP   141
      YMAX = 0.0                                                          OUTP   142
      IF (IPLOT-3) 299, 302, 44                                           OUTP   143
  299 DO 301 MM = 1, II                                                   OUTP   144
      ZZ(MM) = ZZ(NP+MM)                                                  OUTP   145
  301 ZZ(MM) = ZZ(NP+MM)                                                  OUTP   146
C                                                                         OUTP   147
C     PLOT TRANSLATIONS                                                   OUTP   148
C                                                                         OUTP   149
      CALL MAXIN( ZZ, II, YMAX, YMIN )                                    OUTP   150
      CALL SYMBOL( -0.5,   3.9, 0.25, 11HX (CM.)    , 90., 11)            OUTP   151
      GO TO 444                                                           OUTP   152
  302 DO 300 MM = 1, II                                                   OUTP   153
      ZZ(MM) = ZZ(NP*2+MM)                                                OUTP   154
  300 ZZ(MM) = ZZ(NP*2+MM)                                                OUTP   155
      CALL MAXIN( ZZ, II, YMAX, YMIN )                                    OUTP   156
      CALL SYMBOL( -0.5,   3.9, 0.25, 11HY (CM.)    , 90., 11)            OUTP   157
  444 CALL SYMBOL( 1.5,10.2, .25, 11HTRANSLATION, 0.0, 11 )               OUTP   158
      LN = PRMT(2)/1000.                                                  OUTP   159
      IF (YMIN.GT.-0.05. AND. YMAX.LT.0.05 ) DX=100.*(FLOAT(LN)+1.)       OUTP   160
      IF (IPLOT.LE.3) GO TO 355                                           OUTP   161
   44 PRMT(10) = X                                                        OUTP   162
      XF = PRMT(2) - .9999                                                OUTP   163
      IF (X-XF) 8, 7, 7                                                   OUTP   164
    7 II1 = II + 1                                                        OUTP   165
      II2 = II+2                                                          OUTP   166
      WW(II1) = 0.0                                                       OUTP   167
      WW(II2) = 0.0                                                       OUTP   168
      XX(II1) = PRMT(2)                                                   OUTP   169
      XX(II2) = 0.0                                                       OUTP   170
      YY(II1) =0.0                                                        OUTP   171
      YY(II2) =0.0                                                        OUTP   172
```

```
      PP(II1) = 0.0
      PP(II2) = 0.0
      QQ(II1) = 0.0
      QQ(II2) =0.0
      YMIN = -0.15
      DX = 50.0
      YMAX = 0.2
      AMAX = 0.0
      AMIN = 0.0
C
C     SCALING Y-AXIS OF PLOTS #4,5 (X-AXIS IS TIME, IN SEC.)
C
      CALL MAXIN( YY, II, AMAX, AMIN )
      CALL MAXIN( WW, II, AMAX, AMIN )
      CALL MAXIN( PP, II, AMAX, AMIN )
      CALL MAXIN( QQ, II, AMAX, AMIN )
      XAMA = AMAX1( AMAX, ABS(AMIN))
      IF (XAMA.GT.YMAX) YMAX = XAMA*1.05
      IF (XAMA.LT.0.15) YMAX=0.15
      IF (XAMA.LE.0.1) YMAX=0.10
      YMIN = -YMAX
      DY = YMAX/5.0
      IF (PRMT(2).GT.1100.0) DX = 100.
      AA(2) = PRMT(2)
      AA(3) = PRMT(2)
      BB(1) = YMAX
      BB(2) = YMAX
      BB(3) = YMIN
      BB(4) = YMIN
   70 CONTINUE
      PLOTI = IPLOT
C
C     PLOT BOOM OSCILLATIONS
C
      CALL LINE( AA, BB, 4, 1, 0, 3, XMIN, DX, YMIN, DY, 0.08 )
      IN = INT(PRMT(2))/INT(DX)
      DO 2 I = 1, IN
      T = FLOAT(I)
    2 CALL SYMBOL( T, 0., 0.1, 13, 0.0, -1 )
      DO 22 I = 1, 5
      T = FLOAT(2*I) - 1.0
   22 CALL SYMBOL( 0., T, 0.1, 13, 90.0, -1 )
      INN= INT( PRMT(2)/DX*0.5 ) + 1
      DO 4  I = 1, INN
      T = FLOAT(2*I-2) - 0.1
      FPN = 2.*DX*(FLOAT(I)-1.0)
    4 CALL NUMBER( T, 0.05, 0.19, FPN, 0.0, -1 )
      XMAX = PN/DX
      CALL AXIS( 0.0, 0.0, 9HPHI(RAD.),  9, 10.0, 90.0, YMIN, DY, 5. )
      CALL AXIS( XMAX, 0.0, 9HPHI(RAD.), -9, 10.0, 90.0, YMIN, DY, 5. )
      CALL SYMBOL( 2.5, 0.28,0.25, 10HTIME(SEC.), 0., 10 )
      IF (IPLOT.GT.4) GO TO 71
      CALL LINE(XX, YY, II2, 1, 0, 3, XMIN, DX, YMIN, DY, 0.08)
      CALL LINE(XX, WW, II2, 1,20,11, XMIN, DX, YMIN, DY, 0.08)
      GO TO 72
   71 CONTINUE
      CALL LINE ( XX, PP, II2, 1, 0, 3, XMIN, DX, YM[illegible] [illegible]Y, 0.08 )
```

```
      CALL LINE( XX, QQ, II2, 1, 20, 11, XMIN, DX, YMIN, DY, 0.08)
   72 CONTINUE
      T = FLOAT(IN) + 3.0
      CALL PLOT( T, 0.0, -3 )
      IPLOT = IPLOT + 1
      IF (IPLOT.LE.5) GO TO 70
C
C     OPTION-IHARM=0 IF FFT IS TO BE PERFORMED,
C            IHARM=1 IF FFT IS TO BE OMITTED.
C
      IHARM = PRMT(5)
      IF (IHARM.EQ.1) GO TO 34
      AA(1) = 0.0
      AA(2) = 0.0
      AA(3) = 12.0
      AA(4) = 12.0
      BB(1) = 0.0
      BB(2) = 5.0
      BB(3) = 5.0
      BB(4) = 0.0
      T = PRMT(2)/DT
      M = INT(ALOG(T)/ALOG(2.) + 1.E-30)
      II = 2**M
      WRITE(6,66) M, II
   66 FORMAT(/, 1X, *M = *, I3, 7X, *II = *, I5,/)
      CALL PLOT(0.0, 1.0, -3)
      CALL POWER( XX, YY, II, AA, BB, DT )
C
C     OPTIONAL POWER SPECTRAL ANALYSIS
C
      CALL POWER( XX, WW, II, AA, BB, DT )
      CALL POWER( XX, PP, II, AA, BB, DT )
      CALL POWER( XX, QQ, II, AA, BB, DT )
   34 CONTINUE
      MM = 552
      IF (MM.GT.II) MM=II
      AMAX = 0.0
      AMIN = 0.0
      CALL MAXIN( YY, MM, AMAX, AMIN )
      CALL MAXIN( WW, MM, AMAX, AMIN )
      CALL MAXIN( PP, MM, AMAX, AMIN )
      CALL MAXIN( QQ, MM, AMAX, AMIN )
      DO 61 I = 1, MM
      YY(I) = YY(I)/AMAX
      PP(I) = PP(I)/AMAX
      WW(I) = WW(I)/AMAX
      QQ(I) = QQ(I)/AMAX
   61 FREQ(I) = 0.5*(FLOAT(I)-1.0)/(FLOAT(II)*DT)
      WRITE(6,60)
   60 FORMAT(1H1,*POWER SPECTRA OF BOOM OSCILLATIONS*,///,T7,*BOOM 1*, T
     118,*BOOM 2*,T29,*BOOM 3*,T40,*BOOM 4*,T52,*FREQ(CPS)*,T67,*NO.*,/)
      WRITE(6,62) (YY(I), WW(I), PP(I), QQ(I), FREQ(I), I, I=1,MM)
   62 FORMAT(1X, 4F11.5, 1X, F11.5, 6X, I4 )
C
C     OPTIONAL PLOTS OF POWER SPECTRA
C
      CALL SPECPLT( AA, BB, JJ, MM, MMM, FREQ, YY )
```

```
      CALL SPECPLT( AA, BB, JJ, MM, MMM, FREQ, WW )
      CALL SPECPLT( AA, BB, JJ, MM, MMM, FREQ, PP )
      CALL SPECPLT( AA, BB, JJ, MM, MMM, FREQ, QQ )
      II = 0
      YMIN = 0.0
      PRMT(10) = -2.0
      X = PRMT(2)
      DO 501 I=1, 50
C     WRITE(6,502) Y(I)
  501 IF (PRMT(7).EQ.1.0) PUNCH 502, Y(I)
  502 FORMAT(1X, F20.10)
    8 RETURN
      END
```

SUBROUTINE DAMPER   74/74   OPT=1   FTN 4.1+P370   12/20/74 16.33.03.   PAGE   1

```
      SUBROUTINE DAMPER(X, Y, DAM, I, B, PHC)
      DIMENSION Y(1), B(1), PHC(1)
      DIMENSION YMAX(4)
      COMMON /DDD/ KAMP(4), KEY(4)
C
C     IF THERE IS DAMPING, KAMP(I) = 1
C
    2 J = 2*I+2
      K = 2*I+1
      Y4 = Y(22+2*I)
      SIGN = Y(J)/(ABS(Y(J))+1.E-30)
      SN = Y4*Y(J)
      SLOPE = 0.5*ABS( Y4+Y(J) )
C
C     TEST WHETHER SWING RETURNS
C
      IF (SN.LT.0.0.AND.SLOPE.LT.0.001) KEY(I) = 1
      IF (KAMP(I).NE.1) GO TO 55
      IF (SN.LT.0.0.AND.SLOPE.LT.0.001) YMAX(I) = Y(21+2*I)
   55 CONTINUE
C
C     UPDATE MAXIMUM ANGULAR DISPLACEMENT
C
      IF (Y(K).GT. PHC(I)) YMAX(I) = PHC(I)
      IF (Y(K).LT.-PHC(I)) YMAX(I) =-PHC(I)
      KAMP(I) = 2
      IF (KEY(I).NE.1) GO TO 6
      IF (ABS(Y(K)).GT.PHC(I)) GO TO 7
      KAMP(I) = 0
      DIF = ABS(YMAX(I)-Y(K))-2.0*Y(14+I)
      IF (DIF.GT.0.0) B(I) = B(I) - DAM*SIGN
      IF (DIF.GT.0.0) KAMP(I) = 1
      GO TO 7
    6 CONTINUE
      PHI = ABS(Y(K))
      KAMP(I) = 0
      IF (ABS(Y(K)).GT.PHC(I)) KAMP(I) = 2.0
      IF (PHI.GE.Y(14+I). AND. PHI.LT.PHC(I)) KAMP(I) = 1
      IF (PHI.GE.Y(14+I). AND. PHI.LT.PHC(I)) B(I) = B(I) - DAM*SIGN
    7 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE POWER(X, Y, II, AA, BB, DT)
C
C     LAGRANGIAN INTERPOLATION TO GENERATE EQUALLY SPACED TIME SERIES
C     FOR HARMONIC ANALYSIS
C
      DIMENSION X(1), Y(1), AA(1), BB(1),
     1          A(4100), INV(512), S(512)
      MM = 552
      IF (MM.GT.II) MM II
      DO 15 I = 1, II
   16 A(I) = Y(I)
C
C     10-POINT INTERPOLATION
C
      DO 62 K = 6, II
      XK = DT*(FLOAT(K)-1.0)
      IF ( X(K). EQ. XK ) GO TO 62
      C = 0.
      KM = K + 1
   77 KM = KM - 1
      IF (X(KM)-XK) 78, 78, 77
   78 KMAX = KM + 6
      KMIN = KM - 4
      IF (KMAX.GT.II) KMAX=II
      DO 52 I = KMIN, KMAX
      P = 1.
      DO 40 J = KMIN, KMAX
      IF (I-J) 35, 40, 35
   35 P = P*(XK-X(J))/(X(I)-X(J))
   40 CONTINUE
   52 C = C + P*Y(I)
      A(K) = C
   62 CONTINUE
      DO 2 I = 1, II
    2 Y(I) = A(I)
      T = II
      M = INT( ALOG(T)/ALOG(2.0) + 0.0000001 )
C
C     PERFORM FAST FOURIER TRANSFORM ON THE TIME SERIES
C
      CALL RHARM( A, M, S, INV, IFERR )
      DO 3 I=1, MM
    3 Y(I) = A(2*I-1)**2 + A(2*I)**2
      Y(I) = 6.*Y(I)
      RETURN
      END
```

```
      SUBROUTINE SPECPLT( AA, BB, JJ, MM, MMM, FREQ, ABSQ )            SPECPLT   2
      DIMENSION FREQ(1), ABSQ(1), AA(1), BB(1)                         SPECPLT   3
      COMMON/FFT/ NPLOT, NHIS                                          SPECPLT   4
      JJ = JJ + 1                                                      SPECPLT   5
      I = 50                                                           SPECPLT   6
      FMX = 0.95                                                       SPECPLT   7
   63 IF (FREQ(I)-FMX) 65, 64, 64                                      SPECPLT   8
   65 I = I + 1                                                        SPECPLT   9
      IF (I-MM) 63, 64, 64                                             SPECPLT  10
   64 MMM = I                                                          SPECPLT  11
      IF (NPLOT) 65, 18, 66                                            SPECPLT  12
C                                                                      SPECPLT  13
C     PLOT FREQUENCY SPECTRUM OF PHI(I) (I=1,...,4)                    SPECPLT  14
C                                                                      SPECPLT  15
   66 CALL LINE( AA, BB, 5, 1, 0, 3, 0.0, 1.0, 0.0, 1.0, 0.08)         SPECPLT  16
      CALL SYMBOL( 3.3, -3.6, 0.25, 9HFREQ(CPS), 0., 9)                SPECPLT  17
      CALL SYMBOL( 3.5, 8.4, 0.2, 14HPOWER SPECTRUM, 0.0, 14 )         SPECPLT  18
      CALL SYMBOL( 2.1, 8.4, 0.2, 6HPHI( ), 0.0, 6 )                   SPECPLT  19
      AJJ = JJ                                                         SPECPLT  20
      IF (JJ.LT.5) CALL NUMBER( 2.9, 8.4, 0.2, AJJ, 0.0, -1)           SPECPLT  21
      DO 10 I = 1, 10                                                  SPECPLT  22
      T = FLOAT(I)*1.2                                                 SPECPLT  23
   10 CALL SYMBOL( T, 0., 0.1, 13, 0.0, -1 )                           SPECPLT  24
      DO 11 I=1,6                                                      SPECPLT  25
      FPN = FLOAT(2*I-2)*0.1                                           SPECPLT  26
      T = FPN*3.(?)                                                    SPECPLT  27
    9 CALL SYMBOL( 0., T, 0.1, 13, 90.0, -1 )                          SPECPLT  28
   11 CALL NUMBER(-.05, T, 0.2, FPN, 90.0, 1 )                         SPECPLT  29
      DO 12 I = 1, 6                                                   SPECPLT  30
      T = FLOAT(2*I-2)*1.2                                             SPECPLT  31
      FPN = (FLOAT(I)-1.0)/50.0                                        SPECPLT  32
      FPN = FPN*0.5                                                    SPECPLT  33
   12 CALL NUMBER( T, -0.25, 0.2, FPN, 0., 2 )                         SPECPLT  34
      DX = FMX/12.?                                                    SPECPLT  35
      CALL LINE( FREQ, ABSQ,MMM, 1, 0, 3, 0.0, DX, 0.0, .125, 0.08)    SPECPLT  36
      CALL PLOT( 14.24, 0.0, -3 )                                      SPECPLT  37
   18 CONTINUE                                                         SPECPLT  38
      IF (NHIS.NE.1) RETURN                                            SPECPLT  39
C                                                                      SPECPLT  40
C     OPTIONAL HISTOGRAMS OF POWER SPECTRA                             SPECPLT  41
C                                                                      SPECPLT  42
      CALL SPECPA( FREQ, ABSQ, MM, 10HFREQ.(CPS),10HPOWER DEN.,0.,1.,1 ) SPECPLT  43
      RETURN                                                           SPECPLT  44
      END                                                              SPECPLT  45
```

```
      SUBROUTINE BAL( Y, R, DELTA, RDOT, AM, AMO, RO, RHO )
      DIMENSION Y(1), R(1), DELTA(1), RDOT(1)
      Y(12) = 0.0
      Y(14) = 0.0
      DSUM = 0.0
C
C     CONSERVATION OF LINEAR MOMENTA
C
      DO 111 I = 1, 4
      DELLL = DELTA(I) + Y(1)
      K = 2*I+2
      DSUM = DSUM + (AM*RHO*R(I))
      Y(12) = Y(12) + (AM*RHO*R(I))*(-RO*SIN(DELLL)    *Y(2))-(AM*R(I)*
     1(.5*RHO*R(I)**2)*(Y(2)+Y(K))*SIN(DELTA(I)+Y(2*I+1)+Y(1))
     2+(AM*RHO*R(I))*RDOT(I)*COS(DELLL+Y(2*I+1))
      Y(14) = Y(14) + (AM*RHO*R(I))*( RO*COS(DELLL)    *Y(2))+(AM*R(I)*
     1(.5*RHO*R(I)**2)*(Y(2)+Y(K))*COS(DELTA(I)+Y(2*I+1)+Y(1))
     2+(AM*RHO*R(I))*RDOT(I)*SIN(DELLL+Y(2*I+1))
  111 CONTINUE
      Y(12) =-Y(12)/(DSUM+AMO)
      Y(14) =-Y(14)/(DSUM+AMO)
      RETURN
      END
```

```
      SUBROUTINE SPCTRA( X, Y, N, XNAME, YNAME, Y1, Y2, MODE )           SPCTRA    2
C                                                                        SPCTRA    3
C.............THIS SUBROUTINE YIELDS A POINT PLOT OR A SOLID PLOT        SPCTRA    4
C.............DEPENDENT UPON THE WAY "MODE" IS SPECIFIED.                SPCTRA    5
C                                                                        SPCTRA    6
C     X=NAME OF THE X-ARRAY.                                             SPCTRA    7
C     Y=NAME OF THE Y-ARRAY.                                             SPCTRA    8
C     N=NUMBER OF POINTS TO BE PLOTTED.                                  SPCTRA    9
C     XNAME=TITLE INFORMATION FOR THE X-ARRAY, ASSUMES BCD CHARACTERS.   SPCTRA   10
C           USUALLY BE OF THE FORM, 10H TITLE (MUST BE 10OR LESS H).     SPCTRA   11
C     YNAME=TITLE INFORMATION FOR THE Y-ARRAY.                           SPCTRA   12
C     Y1=MINIMUM OF Y ON THE PLOT.                                       SPCTRA   13
C     Y2=MAXIMUM OF Y ON THE PLOT.                                       SPCTRA   14
C     IF Y1=Y2 OR Y1.GT.Y2 THE SUBROUTINE WILL SCALE THE PLOT OF THE DAT SPCTRA   15
C     MODE<  IF MODE=0, A POINT PLOT WILL BE PRODUCED.                   SPCTRA   16
C            IF MODE .NE. 0, A SOLID PLOT WILL BE PRODUCED.              SPCTRA   17
C                                                                        SPCTRA   18
      DIMENSION X(2), Y(2), A(101), AXIS(6)                              SPCTRA   19
       DATA STAR,BLANK/1H*,1H /                                          SPCTRA   20
       DATA DASH,ZI/1H-,1HI/                                             SPCTRA   21
      YMIN = Y1                                                          SPCTRA   22
      YMAX = Y2                                                          SPCTRA   23
      IF (YMIN.NE.YMAX) GO TO 100                                        SPCTRA   24
      YMIN = Y(1)                                                        SPCTRA   25
      YMAX = Y(1)                                                        SPCTRA   26
      DO 10 I = 2, N                                                     SPCTRA   27
      IF (YMAX.LT.Y(I)) YMAX = Y(I)                                      SPCTRA   28
   10 IF (YMIN.GT.Y(I)) YMIN = Y(I)                                      SPCTRA   29
  100 CONTINUE                                                           SPCTRA   30
      ICOUNT = 0                                                         SPCTRA   31
      DY = (YMAX-YMIN) / 5.0                                             SPCTRA   32
      AXIS(1) = YMIN                                                     SPCTRA   33
      DO 12 I = 2, 6                                                     SPCTRA   34
   12 AXIS(I) = AXIS(I-1)+DY                                             SPCTRA   35
      WRITE(6,20) YNAME, XNAME                                           SPCTRA   36
   20 FORMAT(1H1,52X,  9HGRAPH OF    ,A10,4H VS , A10)                   SPCTRA   37
      WRITE (6,22) XNAME, YNAME                                          SPCTRA   38
   22 FORMAT( 2X, A10, 108X, A10)                                        SPCTRA   39
      WRITE (6,23) AXIS                                                  SPCTRA   40
 23   FORMAT (F18.3,5F20.3)                                              SPCTRA   41
      WRITE (6,24) (ZI  , I=1,6 )                                        SPCTRA   42
   24 FORMAT   (14X, 5(A1,19X ),A1)                                      SPCTRA   43
      WRITE (6,25) (DASH, I=1,101)                                       SPCTRA   44
   25 FORMAT (14X, 101A1 )                                               SPCTRA   45
      DELTA = YMAX - YMIN                                                SPCTRA   46
        IZERO = IFIX(-YMIN/DELTA*100.)+1                                 SPCTRA   47
       IF (IZERO.GT.101.OR.IZERO.LT.1) IZERO=1                           SPCTRA   48
      DO 354 M = 1, N                                                    SPCTRA   49
      DO 73 K =2,100                                                     SPCTRA   50
   73 A(K) = BLANK                                                       SPCTRA   51
       A(1)=ZI                                                           SPCTRA   52
       A(IZERO) = ZI                                                     SPCTRA   53
        A(101) = ZI                                                      SPCTRA   54
   35 CONTINUE                                                           SPCTRA   55
      IY = IFIX(((Y(M)-YMIN)/DELTA*100.) + SIGN(.5000001,Y(M))) + 1      SPCTRA   56
      IF (IY.LT.1) GO TO 300                                             SPCTRA   57
      IF (IY.GT.101) IY=101                                              SPCTRA   58
```

```
      IF ( MODE.GT.0) GO TO 900
       A(IY) = STAR
      GO TO 300
  900 CONTINUE
      DO 880 L =1, IY
 880    A(L) = STAR
  300 CONTINUE
      WRITE (6,8) X(M),A,Y(M)
    8 FORMAT (1X, F10.6, 3X, 101A1, 2X, 1PE14.5)
  354 CONTINUE
      WRITE (6,25) (DASH, I=1,101)
      WRITE (6,24) (ZI  , I=1,6)
      WRITE (6,23) AXIS
      WRITE (6,22) XNAME, YNAME
      WRITE (6,21) YNAME, XNAME
   21 FORMAT (//,52X,  9HGRAPH OF    ,A10,4H VS , A10)
      RETURN
      END
```

```
SUBROUTINE MAXIN      74/74    OPT=1                        FTN 4.1+P378        12/20/74  16.33.22.       PAGE     1

              SUBROUTINE MAXIN(A, MM, AMAX, AMIN )                          MAXIN       2
              DIMENSION A(1)                                                MAXIN       3
              DO 1 I=1, MM                                                  MAXIN       4
              IF (A(I).GT.AMAX) AMAX=A(I)                                   MAXIN       5
            1 IF (A(I).LT.AMIN) AMIN=A(I)                                   MAXIN       6
              RETURN                                                        MAXIN       7
              END                                                           MAXIN       8
```

Sample for Program Satedyn follows

```
INPUT CARDS :-

1
    20.00     20.00     20.00     20.00
   200.00-1 1-1 1
     0.00      0.00      0.00      0.00
1    3.0000 1 1
  1050.00
```

```
DAMPER ACTION SYMBOLS ---
                          0 = NO DAMPING, ANGLE TOO SMALL.
                          1 = DAMPER IN ACTION.
                          2 = NO DAMPING, ANGLE TOO LARGE.

TENSION SIGN SYMBOLS ----
                          RATIO OF TENSION(PHI,X,Y) TO TENSION(PHI=0,X=0,Y=0)
```

INPUT DATA FOR 1975 SATELLITE PROBLEM ....DATA SET 1

INITIAL WIRE LENGTH	=	20.00000 FT.	=	609.60000 CM.	(1)
INITIAL WIRE LENGTH	=	20.00000 FT.	=	609.60000 CM.	(2)
INITIAL WIRE LENGTH	=	20.00000 FT.	=	609.60000 CM.	(3)
INITIAL WIRE LENGTH	=	20.00000 FT.	=	609.60000 CM.	(4)
FINAL WIRE LENGTH	=	10.00000 FT.	=	304.80000 CM.	(1)
FINAL WIRE LENGTH	=	30.00000 FT.	=	914.40000 CM.	(2)
FINAL WIRE LENGTH	=	10.00000 FT.	=	304.80000 CM.	(3)
FINAL WIRE LENGTH	=	30.00000 FT.	=	914.40000 CM.	(4)
RATE OF DEPLOYMENT	=	-.05000 FT./SEC.	=	-1.52400 CM./SEC.	(1)
RATE OF DEPLOYMENT	=	.05000 FT./SEC.	=	1.52400 CM./SEC.	(2)
RATE OF DEPLOYMENT	=	-.05000 FT./SEC.	=	-1.52400 CM./SEC.	(3)
RATE OF DEPLOYMENT	=	.05000 FT./SEC.	=	1.52400 CM./SEC.	(4)
MOMENT OF INERTIA OF HUB	=	3702.40000 LB.FT.2	=	1560195539.3 GM.CM.2.	
TOTAL MOMENT OF INERTIA	=	7820.4430 LB.FT.2	=	3295546034.8 GM.CM.2.	
TIP MASS	=	2.00000 LB.	=	907.18480 GM.	
MASS OF HUB	=	490.90000 LB.	=	222668.50916 GM.	
RADIUS OF HUB	=	2.57000 FT.	=	81.38160 CM.	
MASS DENSITY OF BOOM	=	.00043 LB./FT.	=	.00639 GM./CM.	
PERIOD OF DEPLOYMENT	=	200.00000 SEC.			
INITIAL ANGULAR VELOCITY	=	.74645 RAD/SEC		(FINAL ANG. VEL.: 3.00000 RPM)	
KPHI	=	.62373 LB.FT.2/SEC.RAD.	=	10000.00000 GM.CM.2/SEC.RAD.	
S	=	1.00000 LB.FT.2/RAD.SEC.2	=	400066.8 GM.CM.2/RAD.SEC.2	

(CONVERSION FACTORS : 1 LB.=453.5924 GM., 1 FT.=30.48 CM.)

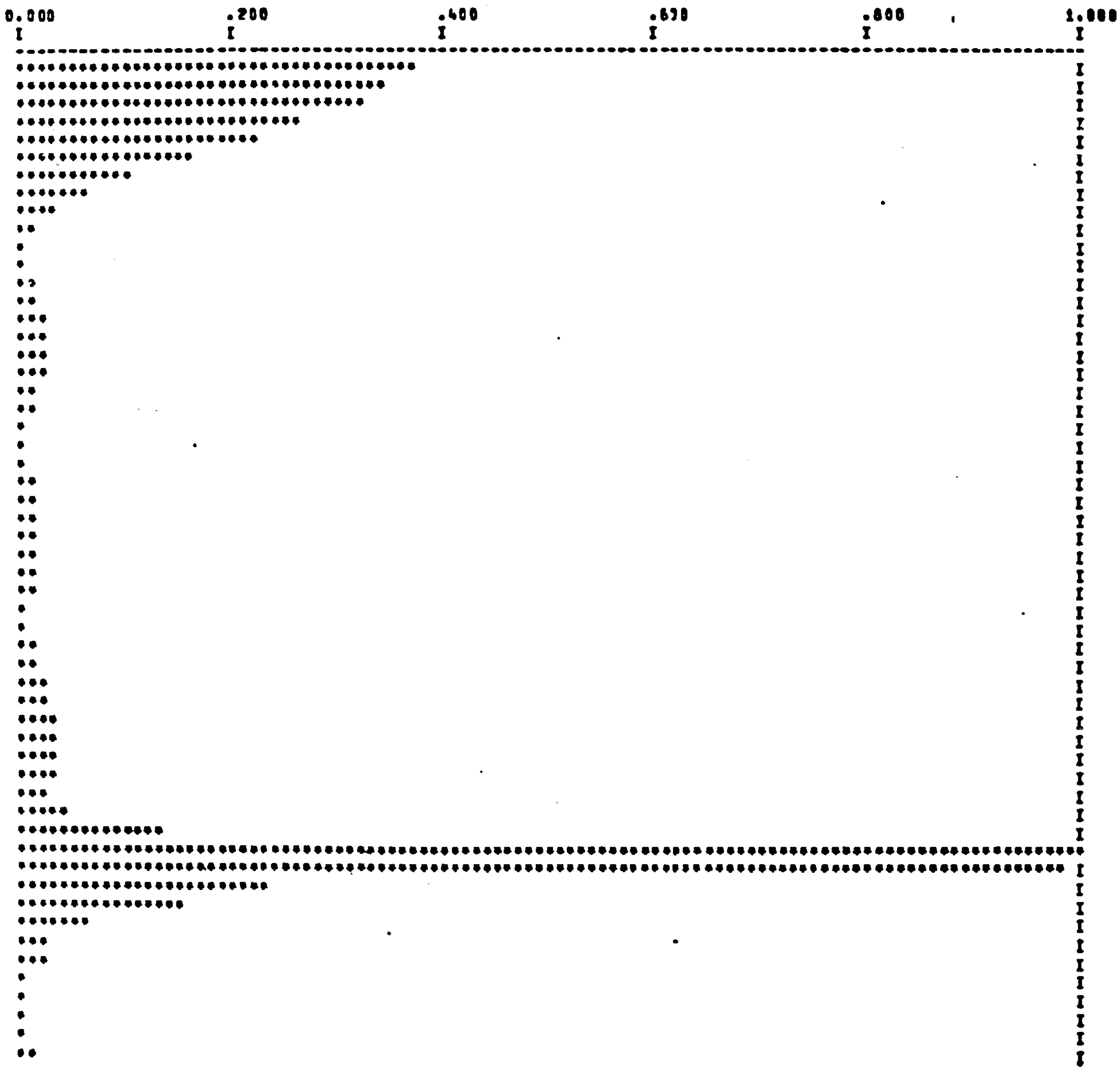
OUTPUT DATA : ....

TIME SEC.	PHI 1 RAD.	PHI 2 RAD.	PHI 3 RAD.	PHI 4 RAD.	HUB VEL RAD/SEC	X CM.	X-VEL CM/SEC	Y CM.	Y-VEL CM/SEC	DAMPERS ON-OFF	TENSIONS RATIOS	NO.
0.00	0.0000	0.0000	0.0000	0.0000	.346452	0.0000	-.0000	0.0000	.0000	0 0 0 0	1.0 1.0 1.0 1.0	1
1.00	.0009	-.0003	.0003	-.0009	.346452	.0000	.0000	-.0000	-.0000	0 0 0 0	1.0 1.0 1.0 1.0	2
2.00	.0035	-.0034	.0035	-.0034	.346452	.0000	.0000	-.0000	-.0000	0 0 0 0	1.0 1.0 1.0 1.0	3
3.00	.0078	-.0075	.0073	-.0076	.346454	.0000	.0000	-.0000	-.0000	0 0 0 0	1.0 1.0 1.0 1.0	4
4.00	.0137	-.0134	.0137	-.0134	.346460	.0000	.0000	-.0000	.0000	0 0 0 0	1.0 1.0 1.0 1.0	5
5.00	.0211	-.0207	.0211	-.0207	.346470	.0000	.0000	.0000	.0000	0 0 0 0	1.0 1.0 1.0 1.0	6
6.00	.0300	-.0292	.0301	-.0292	.346488	.0000	.0000	.0000	.0000	0 0 0 0	1.0 1.0 1.0 1.0	7
7.00	.0401	-.0393	.0401	-.0390	.346516	.0000	-.0000	.0000	.0000	0 0 0 0	1.1 .9 1.1 .9	8
8.00	.0513	-.0493	.0513	-.0498	.346559	.0000	-.0000	.0000	.0000	1 0 1 0	1.1 .9 1.1 .9	9
9.06	.0641	-.0622	.0641	-.0622	.346634	-.0000	-.0000	.0000	.0000	1 1 1 1	1.1 .9 1.1 .9	10
10.19	.0782	-.0769	.0782	-.0760	.346706	-.0000	-.0000	.0000	-.0000	1 1 1 1	1.1 .9 1.1 .9	11
11.44	.0943	-.0913	.0943	-.0919	.346800	-.0000	-.0000	.0000	-.0000	1 1 1 1	1.1 .9 1.1 .9	12
12.44	.1071	-.1045	.1071	-.1046	.346882	-.0000	-.0000	.0000	-.0000	1 1 1 1	1.1 .9 1.1 .9	13
13.47	.1201	-.1177	.1201	-.1177	.346950	-.0000	-.0000	-.0000	-.0000	2 1 2 1	1.1 .9 1.1 .9	14
14.50	.1327	-.1303	.1327	-.1303	.347030	-.0000	.0000	-.0000	-.0000	2 2 2 2	1.1 .9 1.1 .9	15
15.53	.1457	-.1436	.1457	-.1436	.347115	-.0000	.0000	-.0000	-.0000	2 2 2 2	1.1 .9 1.1 .9	16
16.53	.1564	-.1547	.1564	-.1547	.347178	-.0000	.0000	-.0000	-.0000	2 2 2 2	1.1 .9 1.1 .9	17
17.53	.1560	-.1641	.1660	-.1649	.347222	.0000	.0000	-.0000	.0000	2 2 2 2	1.0 .9 1.0 .9	18

POWER SPECTRA OF BOOM OSCILLATIONS

BOOM 1	BOOM 2	BOOM 3	BOOM 4	FREQ(CPS)	NO.
.53343	.36705	.53344	.36705	0.00000	1
.48622	.33576	.48622	.33576	.00049	2
.46955	.32212	.46955	.32212	.00098	3
.38065	.26265	.38065	.26265	.00146	4
.31272	.21708	.31272	.21708	.00195	5
.22384	.15571	.22384	.15571	.00244	6
.14485	.10244	.14485	.10244	.00293	7
.05733	.06062	.05733	.06052	.00342	8
.03478	.02547	.03478	.02547	.00391	9
.01385	.00940	.01385	.00940	.00439	10
.00026	.00081	.00026	.00081	.00488	11
.00114	.00156	.00114	.00156	.00537	12
.01259	.00956	.01259	.00956	.00586	13
.01713	.01357	.01713	.01357	.00635	14
.02932	.02138	.02932	.02138	.00684	15
.02829	.02085	.02829	.02085	.00732	16
.02663	.01943	.02663	.01943	.00781	17
.02158	.01522	.02158	.01522	.00830	18
.01025	.00755	.01025	.00755	.00879	19
.00682	.00504	.00682	.00504	.00928	20
.00019	.00032	.00019	.00032	.00977	21
.00006	.00020	.00006	.00020	.01025	22
.00546	.00415	.00546	.00415	.01074	23
.00746	.00559	.00746	.00559	.01123	24
.01724	.01326	.01724	.01326	.01172	25
.01638	.01234	.01638	.01234	.01221	26
.01833	.01432	.01833	.01432	.01270	27
.01581	.01243	.01581	.01243	.01318	28
.00832	.00705	.00832	.00705	.01367	29
.00657	.00518	.00657	.00518	.01416	30
.00026	.00059	.00026	.00059	.01465	31
.00006	.00015	.00006	.00015	.01514	32
.00596	.00561	.00596	.00561	.01563	33
.00585	.00540	.00585	.00540	.01611	34
.02373	.02157	.02373	.02157	.01660	35
.02331	.02063	.02331	.02063	.01709	36
.03891	.03352	.03891	.03352	.01758	37
.04029	.03306	.04029	.03306	.01807	38
.03620	.02864	.03620	.02864	.01855	39
.04073	.02901	.04073	.02901	.01904	40
.01463	.01895	.01463	.01895	.01953	41
.03426	.04158	.03426	.04158	.02002	42
.05079	.12869	.05079	.12869	.02051	43
.49395	1.00000	.49395	1.00000	.02100	44
.46652	.98290	.43652	.98290	.02148	45
.13396	.22923	.13396	.22923	.02197	46
.07591	.14655	.07591	.14655	.02246	47
.02896	.05721	.02896	.05721	.02295	48
.01713	.02391	.01713	.02391	.02344	49
.00842	.01526	.00842	.01526	.02393	50
.00487	.00048	.00487	.00048	.02441	51
.00349	.00178	.00349	.00178	.02490	52
.00207	.00274	.00207	.00274	.02539	53
.00134	.00061	.00134	.00061	.02588	54
.00076	.00475	.00076	.00675	.02637	55

GRAPH OF POWER DEN. VS FREQ.(CPS)



FREQ.(CPS)	POWER DEN.
0.000000	3.67055E-01
.000488	3.35759E-01
.000977	3.22120E-01
.001465	2.62647E-01
.001953	2.17080E-01
.002441	1.55705E-01
.002930	1.02439E-01
.003418	6.06216E-02
.003906	2.54659E-02
.004395	9.39640E-03
.004883	8.09552E-04
.005371	1.56428E-03
.005859	9.56243E-03
.006348	1.35671E-02
.006836	2.13839E-02
.007324	2.08456E-02
.007813	1.94259E-02
.008301	1.52162E-02
.008789	7.55038E-03
.009277	5.03765E-03
.009765	3.16595E-04
.010254	1.98303E-04
.010742	4.14626E-03
.011230	5.59163E-03
.011719	1.32631E-02
.012207	1.23446E-02
.012695	1.43201E-02
.013184	1.24260E-02
.013672	7.05287E-03
.014160	5.18289E-03
.014649	5.87130E-04
.015137	1.51111E-04
.015625	5.60872E-03
.016113	5.39940E-03
.016602	2.15718E-02
.017090	2.06318E-02
.017578	3.35157E-02
.018066	3.30621E-02
.018555	2.86376E-02
.019043	2.90139E-02
.019531	1.89450E-02
.020020	4.15806E-02
.020508	1.28691E-01
.020996	1.00000E+00
.021484	9.82900E-01
.021973	2.29226E-01
.022461	1.46550E-01
.022949	5.72060E-02
.023438	2.39080E-02
.023925	1.52587E-02
.024414	4.79614E-04
.024902	1.77719E-03
.025391	2.74418E-03
.025879	6.86015E-04
.026367	6.75262E-03

Program Synharm Listing follows.

```
      PROGRAM SYNHARM(INPUT, OUTPUT, TAPE5=INPUT, TAPE6=OUTPUT)
C
C     PROGRAM NO.7, FOR SATELLITE DYNAMICS.
C     HARMONIC OSCILLATIONS OF SATELLITE-WIRE BOOM SYSTEM IN OR OUT OF
C     SPIN PLANE, SIMULATED BY FOURIER SYNTHESIS.
C
C     PREPARED BY LOGICON, INC., DEC. 1974,
C     FOR THE ANALYSIS AND SIMULATION BRANCH, AIR FORCE CAMBRIDGE
C     RESEARCH LABORATORIES, L. G. HANSCOM FIELD, BEDFORD, MASS.
C
C     THIS PROGRAM CALLS THE FOLLOWING IBM SCIENTIFIC SUBROUTINES ( 5TH
C     EDITION, 1970) :-
C     PHARM
C     HARM
C     SIMQ
C
      COMMON PN, DT, AJJJ, IPLOT, IO
      DIMENSION OM(7), B(7,7), P(7), V(7), D(7,7), DELTA(4)
      DIMENSION A(1028), FREQ(512), ABSQ(512), INV(128), SQ(128)
      DIMENSION F(7,1024), X(1024), Y(1024), PROGID(3)
      DIMENSION XX(400), YY(400), TH(2, 400)
      EQUIVALENCE (A,Y)
C
C     INPUT DATA
C
C     FIXED SATELLITE PARAMETERS :-
C     AM0 = HUB MASS (LB.)
C     AM = TIP MASS (LB.)
C     R0 = HUB RADIUS (FT.)
C     RHO = WIRE MASS DENSITY (LB./FT.)
C     S = HINGE POINT SPRING CONSTANT (LB.FT.FT./SEC.SEC.)
C     OMEGA0 = HUB SPIN RATE (RAD./SEC.)
C     AIN = MOMENT OF INERTIA OF HUB (IN-PLANE) --- 115.F SLUG FT.**2
C     AOUT = MOMENT OF INERTIA OF HUB (OUT-OF-PLANE) -- 80 SLUG FT.**2
C
C     PN = SIMULATION PERIOD (SEC.)
C     MM = NO. OF FREQUENCY POINTS TO BE PRINTED OUT.
C     IT = TOTAL NUMBER OF POINTS FOR FAST FOURIER TRANSFORM IN SUBROU-
C     TINE PHARM.
C
      PI = 4.*ATAN(1.)
      TWOPI = 2.*PI
      AM0 = 490.9
      AM = 2.0
      R0 = 2.67
      RHO= 0. 0635
      S  = 1.0
      M = 3
      IT = 2*2**M
      PN = IT
      PN = 600.
      MM = 100
      M11 = MM
      PROGID(1) = 10HLOGICON
      PROGID(2) = 10H   LAI
      PROGID(3) = 10H  HARMONIC
      CALL PLTID( PROGID, 199.99, 12.0, 0.95 )
```

```
      KLM = 4
      READ (5,15) NO
      DO 333 JJJ = 1, NO
      AJJJ = JJJ
C
C     READ INPUTS : ANGULAR VELOCITY(RAD./SEC.), BOOM LENGTHS(FT.),
C     INITIAL POSITIONS P(I) AND VELOCITIES V(I) OF THE 7 GENERALIZED
C     COORDINATES :-
C
C     IN-PLANE VARIABLES :-
C     P(1),..P(4) = PHI(1),...PHI(4)
C     P(5) = THETA(T)-OMEGA0*T
C     P(6) = X
C     P(7) = Y
C
C
C     EQUAL BOOM LENGTH ASSUMED -- R (FT.)
C
      READ(5,111) OMEGA0, R, IO
  111 FORMAT( 2F10.5, I2)
      AA = AM*R**2 + RHO*R**3/3.0
      DD = AM*R + RHO*R**2/2.0
      BB = AA + DD*DD
      AMT= AM0 + 4.0*AM
      QQ = BB*OMEGA0**2 + S
      PP = RD*DD*OMEGA0**2 + S
      WRITE(6,50)
      WRITE(6,15) NO
      WRITE(6,111) OMEGA0, R, IO
   50 FORMAT(1H1, * INPUT CARDS :-*,/)
      READ(5,2) (P(I), I=1, 7)
      WRITE(6,2) (P(I), I=1, 7)
      READ(5,2) (V(I), I=1, 7)
      WRITE(6,2) (V(I), I=1, 7)
    2 FORMAT(7F10.5)
      READ (5,15) IFFT, IM, ISPEC
      WRITE(6,15) IFFT, IM, ISPEC
   15 FORMAT(3I2)
      WRITE(6,4)
    4 FORMAT(1H1)
C
C     V(5) AND V(7) ARE COMPUTED BY BALANCING X AND Y MOMENTA
C
      V(6) = 0.0
      V(7) = 0.0
      DO 222 I = 1, 4
      DELTA(I) = (FLOAT(I) - 1.0)*0.5*PI
      V(5) = V(6) + DD*(OMEGA0+V(I))*SIN(DELTA(I) + P(I))
  222 V(7) = V(7) - DD*(OMEGA0+V(I))*COS(DELTA(I) + P(I))
      V(5) = V(5)/AMT
      V(7) = V(7)/AMT
C
C     OPTIONS :-    IO = 0  IF SIMULATION IS FOR OUT-OF-PLANE DYNAMICS.
C                   IO = 1  IF SIMULATION IS FOR IN-PLANE DYNAMICS.
C
      IF(IO) 88, 87, 88
   87 CALL SYMBOL( 0.5, 5.0, 0.11, 21HOUT-OF-PLANE DYNAMICS, 0., 21 )
```

```
          GO TO 99                                                     SYNHARM  116
       88 CALL SYMBOL( 0.5, 6.0, 0.11, 17HIN-PLANE DYNAMICS, 0.0, 17 )  SYNHARM  117
          AIN = 115.7*32.0                                             SYNHARM  118
          CC = AIN  + (2.0*AM*RO*DD + AA + (AM+PHO*R)*RO**2)*4.0       SYNHARM  119
C                                                                      SYNHARM  120
C         INPLANE B MATRIX                                             SYNHARM  121
C                                                                      SYNHARM  122
          B(5,1) = 1.0/SQRT(CC)                                        SYNHARM  123
          B(6,2) = 1.0/SQRT(AMT)                                       SYNHARM  124
          B(7,3) = B(6,2)                                              SYNHARM  125
          B(1,4) = 0.5/SQRT(AA)                                        SYNHARM  126
          B(2,4) = -B(1,4)                                             SYNHARM  127
          B(3,4) = B(1,4)                                              SYNHARM  128
          B(4,4) =-B(1,4)                                              SYNHARM  129
          B(1,5) = 0.5/SQRT(AA-2.0*DD**2/AMT)                          SYNHARM  130
          B(2,5) = B(1,5)                                              SYNHARM  131
          B(3,5) =-B(1,5)                                              SYNHARM  132
          B(4,5) =-B(1,5)                                              SYNHARM  133
          B(6,5) = 2.0*DD/AMT*B(1,5)                                   SYNHARM  134
          B(7,5) =-B(6,5)                                              SYNHARM  135
          B(1,6) = B(1,5)                                              SYNHARM  136
          B(2,6) = B(2,5)                                              SYNHARM  137
          B(3,6) = B(3,5)                                              SYNHARM  138
          B(4,6) = B(4,5)                                              SYNHARM  139
          B(6,6) = B(7,5)                                              SYNHARM  140
          B(7,6) =-B(6,6)                                              SYNHARM  141
          B(1,7) = 0.5/SQRT(AA-4.0*BB**2/CC)                           SYNHARM  142
          B(2,7) = B(1,7)                                              SYNHARM  143
          B(3,7) = B(1,7)                                              SYNHARM  144
          B(4,7) = B(1,7)                                              SYNHARM  145
          B(5,7) = B(1,7)*(-4.0*BB/CC)                                 SYNHARM  146
          OM(4) = SQRT(PP/(AA*(1.0-2.0*DD**2/AMT/AA)))                 SYNHARM  147
          OM(5) = SQRT(PP/AA)                                          SYNHARM  148
          OM(6) = OM(4)                                                SYNHARM  149
          OM(7) = SQRT(PP/(AA*(1.0-4.0*BB**2/(AA*CC))))                SYNHARM  150
          WRITE(6,65) JJJ                                              SYNHARM  151
       65 FORMAT(1X, *IN-PLANE HARMONIC OSCILLATIONS OF SATELLITE S3-2 ----- SYNHARM  152
         1INPUT INFORMATION *, T135, *DATA SET*, I3, //)               SYNHARM  153
          GO TO 92                                                     SYNHARM  154
C                                                                      SYNHARM  155
C                                                                      SYNHARM  156
       90 AOUT = 80.0*32.0                                             SYNHARM  157
          CC = AOUT + (2.0*AM*RO*DD + AA + (AM+PHO*R)*RO**2)*2.0       SYNHARM  158
C                                                                      SYNHARM  159
C         OUT-OF-PLANE B MATRIX                                        SYNHARM  160
C                                                                      SYNHARM  161
          B(7,1) = AMT**(-0.5)                                         SYNHARM  162
          B(6,2) = CC**(-0.5)                                          SYNHARM  163
          B(5,3) = B(6,2)                                              SYNHARM  164
          B(4,5) = (2.0*(AA-2.0*BB**2/CC))**(-0.5)                     SYNHARM  165
          B(2,5) = B(4,5)                                              SYNHARM  166
          B(3,4) = B(4,5)                                              SYNHARM  167
          B(1,4) = B(4,5)                                              SYNHARM  168
          B(2,6) = 0.5*AA**(-0.5)                                      SYNHARM  169
          B(3,6) = B(2,6)                                              SYNHARM  170
          B(4,6) = -B(2,6)                                             SYNHARM  171
          B(1,6) = -B(2,6)                                             SYNHARM  172
```

```
      B(4,7) = 0.5*(AA-4.0*DD**2/AMT )**(-0.5)                         SYNHARM   173
      B(3,7)  = B(4,7)                                                  SYNHARM   174
      B(2,7) =-B(4,7)                                                   SYNHARM   175
      B(1,7) =-B(4,7)                                                   SYNHARM   176
      B(5,4) =-2.0*BO*B(1,4)/CC                                         SYNHARM   177
      B(6,3) = B(5,4)                                                   SYNHARM   178
      B(7,7) = 4.0*DD/AMT*B(4,7)                                        SYNHARM   179
      OM(4) = SQRT(DD/(AA-2.0*BO**2/CC))                                SYNHARM   180
      OM(5) = OM(4)                                                     SYNHARM   181
      OM(6)  = SQRT(DD/AA)                                              SYNHARM   182
      OM(7) = SQRT(DD/(AA-4.0*DD**2/AMT))                               SYNHARM   183
      WRITE (6,66) JJJ                                                  SYNHARM   184
   66 FORMAT(1X, *OUT-OF-PLANE OSCILLATIONS OF SATELLITE 1975---- INPU  SYNHARM   185
     1T INFORMATION*, T135, *DATA SET*, I3, //)                         SYNHARM   186
   92 CALL PLOT( 5.0, 0.0, -3 )                                         SYNHARM   187
      DO 20 I = 1, 7                                                    SYNHARM   188
      DO 20 J = 1, 7                                                    SYNHARM   189
   20 F(I,J) = B(I,J)                                                   SYNHARM   190
      CPS = OMEGAD/TWOPI                                                SYNHARM   191
      WRITE(6,5) R, OMEGAD, CPS                                         SYNHARM   192
    5 FORMAT(T9,  4HR =, F12.5, 1X, 3HFT.,/,1X, 9HOMEGAD  =, F12.5,1X,  SYNHARM   193
     1 9HRAD./SEC.,2X,1H=,F11.5,1X,4HCPS.)                              SYNHARM   194
      DO 8 I = 1, 7                                                     SYNHARM   195
      CPS = OM(I)/TWOPI                                                 SYNHARM   196
    8 WRITE(6,6) I, OM(I), CPS                                          SYNHARM   197
    6 FORMAT(1X, 6HOMEGA(, I1, 2H)=, F12.5, T24, *RAD./SEC.  =*, F11.5, SYNHARM   198
     11X, 4HCPS.)                                                       SYNHARM   199
      WRITE (6,9)                                                       SYNHARM   200
    9 FORMAT(///, 1X, 14HB MATRIX ......,/)                             SYNHARM   201
      WRITE(6,10) ((B(I,J), J=1,7), I=1,7)                              SYNHARM   202
   10 FORMAT(1X, 7F12.5)                                                SYNHARM   203
      WRITE(6,230)                                                      SYNHARM   204
  230 FORMAT(///, 1X, *INITIAL P MATRIX .....*,/)                       SYNHARM   205
      DO 271 I = 1, 7                                                   SYNHARM   206
  271 WRITE(6,17) P(I)                                                  SYNHARM   207
   17 FORMAT(12X, F12.5)                                                SYNHARM   208
      WRITE(6,300)                                                      SYNHARM   209
  300 FORMAT(///,1X, *INITIAL V MATRIX .....*,/)                        SYNHARM   210
      DO 331 I = 1, 7                                                   SYNHARM   211
  331 WRITE(6,17) V(I)                                                  SYNHARM   212
      WRITE(6,55)                                                       SYNHARM   213
   55 FORMAT(1H1, *OUTPUTS OF SIMULATION -----*, /)                     SYNHARM   214
      IF (ID) 41, 42, 41                                                SYNHARM   215
   41 WRITE(6,45)                                                       SYNHARM   216
   45 FORMAT(///, 1X, 6HT(SEC), T13, 6HPHI(1), T23, 6HPHI(2), T33, 6HPHI SYNHARM   217
     1(3), T43, 6HPHI(4), T53, 6HTHETA*, T64, 6H    X*, T75, 6H    Y*,/) SYNHARM   218
      WRITE(6,36) IT, (P(I), I=1,7 )                                    SYNHARM   219
   36 FORMAT(1X, T4, 3X, 5F10.5, 2F12.5)                                SYNHARM   220
      GO TO 212                                                         SYNHARM   221
   42 WRITE(6,44)                                                       SYNHARM   222
   44 FORMAT(///, 1X, 6HT(SEC), T13, 6HPSI(1), T23, 6HPSI(2), T33, 6HPSI SYNHARM   223
     1(3), T43, 6HPSI(4), T53, 6HTHETA1, T63, 6HTHETA2, T73, 6H     Z,/) SYNHARM   224
      WRITE(6,3) IT, (P(I), I = 1, 7 )                                  SYNHARM   225
  212 CONTINUE                                                          SYNHARM   226
C                                                                       SYNHARM   227
C     SOLVE FOR COEFFICIENTS OF COSINES (IN FOURIER SERIES).            SYNHARM   228
C                                                                       SYNHARM   229
```

```
      CALL SIMQ(B, P, 7, KS)
      DO 7 I = 1, 7
      DO 7 J = 1, 7
    7 B(I,J) = B(I,J)*OM(J)
C
C     SOLVE FOR COEFFICIENTS OF SINES   (IN FOURIER SERIES).
C
      CALL SIMQ(B, V, 7, KS)
      DO 22 I = 1, 7
      DO 22 J = 1, 7
   22 B(I,J) = B(I,J)
C
C     CONSTRUCT FOURIER TIME SERIES OF THE COORDINATES.
C
      DO 12 IT = 1, II
      T = FLOAT(IT)
      X(IT) = T
      DO 11 I = 1, 7
      DO 11 J = 1, 7
   11 F(I,IT)= F(I,IT) + P(J)*B(I,J)*COS(OM(J)*T)
     1                 + V(J)*B(I,J)*SIN(OM(J)*T)
      IF(ID.EQ.0) WRITE(6,3) IT, (F(I,IT), I=1,7)
      IF(ID.EQ.1) WRITE(6,36)IT, (F(I,IT), I=1,7)
   12 CONTINUE
      WRITE(6,35)
   35 FORMAT(///, 1X, *P AND V (EQUATION 11) :-*, //)
      WRITE(6,10) (P(I), I=1,7)
      WRITE(6,10) (V(I), I=1,7)
    3 FORMAT( 1X, I4, 3X, 7F13.5 )
C
C     (XX, YY) ARE COORDINATES OF PRECESSION CONICS.
C
      PRINT 27
   27 FORMAT(1H1,2X,*SEC.*,14X,5HTHETA1, 14X, 6HTHETA2, 14X, 5HXCONIC,
     114X, 5HYCONIC, /)
      T = -.0525
      IMAX = 400
      DO 24 IT = 1, IMAX
      T = .0525 + T
      DO 25 I = 5, 6
      L = I - 4
      DO 25 J = 1, 7
   25 TH(L,IT) = TH(L,IT) + P(J)*B(I,J)*COS(OM(J)*T)
     1                    + V(J)*B(I,J)*SIN(OM(J)*T)
      XN = TAN(TH(1,IT))
      YN = TAN(TH(2,IT))
      XX(IT) = XN*COS(OMEGAD*T)-YN*SIN(OMEGAD*T)
      YY(IT) = XN*SIN(OMEGAD*T)+YN*COS(OMEGAD*T)
   24 WRITE(6,26) ( T, TH(1,IT), TH(2,IT), XX(IT), YY(IT) )
   26 FORMAT(1X, F7.4, F13.5, 3F20.5)
C
C     PLOT PRECESSION-CONICS (DUE TO OUT-OF-PLANE COUPLED OSCILLATIONS)
C
      CALL AXIS( 0.0, 5.0, 1H , -1, 10.0, 0., -0.4, .08, 5.)
      CALL AXIS( 5.0, 0.0, 1H , -1, 10.0,90., -0.4, .08, 5.)
      CALL PLOT( 4.5, 0., -3 )
      CALL SYMBOL( 0.45, 9.0, 0.22, 1HY, 90., 1 )
```

```
      CALL SYMBOL( 4.5, 4.65, 0.22, 1HX, 0., 1)
      CALL LINE (XX, YY, 397, 1, 0, 3, -.004, .008, -.04, .008, .08 )
      CALL PLOT( 0.0, 0., -3)
      CALL SYMBOL( 0.5, 9.0, 0.11, 20HFOURIER SYNTHESIS --, 0., 20)
      CALL SYMBOL( 0.5, 8.5, 0.11, 21HHARMONIC OSCILLATIONS, 0., 21 )
      CALL SYMBOL( 0.5, 8.0, 0.11, 10HNO DAMPING, 0., 10 )
      CALL SYMBOL( 0.5, 7.5, 0.11, 13HNO DEPLOYMENT, 0., 13 )
      CALL PLOT( 5.0, 0.0, -3)
C
C     OPTION -- IFFT = 0 : WITHOUT FFT ANALYSIS
C                    = 1 : WITH FFT ANALSIS
C
      IF (IFFT) 777, 777, 656
C
C     PERFORM FAST FOURIER TRANSFORM ON THE TIME SERIES
C
  656 DT = 1.0
      DO 72 KK = 1, KLM
      WRITE(6,1) KK, KK
    1 FORMAT( 1H1, *HARMONIC ANALYSIS OF OUT-OF-PLANE OSCILLATION ------
     1  VARIABLE *, I1, ///, 1X, T2, *VARIABLE*, I1, T17, *AK*,T27,*BK*,
     2T35, *ABS. SQ.*, T45, *FREQ(CPS)*, T60, *TIME(SEC.)*, /)
      DO 710   = 1, II
  710 A(J) = F(KK,J)
      CALL RHARM( A, M, SO, INV, IFERR )
      AMAX = 0.0
      DO 33 I=1, MM
      ABSQ(I) = A(2*I-1)**2 + A(2*I)**2
      IF (ABSQ(I).GT.AMAX) AMAX = ABSQ(I)
   33 CONTINUE
      DO 61 I = 1, MM
      ABSQ(I) = ABSQ(I)/AMAX
      FREQ(I) =      (FLOAT(I)-1.0)/(FLOAT(II)*DT)
   61 WRITE(6,60)F(KK,I),A(2*I-1),A(2*I), ABSQ(I), FREQ(I), I
   60 FORMAT( 1X, 5F10.5, 6X, I4)
      IF (ISPEC) 70, 71, 70
   71 CALL SPCTRA(FREQ,ABSQ,MM, 10HFREQ.(CPS),10HPOWER DEN.,0.,1.,1 )
   70 CONTINUE
C
C     OPTION -- IM = 0 : FIXED YMAX=0.15, YMIN=-0.15
C                  = 1 : AUTOMATIC YMAX, YMIN.
C
      II = 600.
  777 DO 77 M = 1, 7
      J = ((-1)**M+1)*10
      DO 54 L = 1, II
      Y(L) = F(M,L)
   54 F(M,L) = 0.0
   77 CALL SIM(X, Y, IM, J, II )
      MM = 200
      II = 512
      KLM = 7
      IPLOT = 0
  332 CONTINUE
      STOP
      END
```

```
      SUBROUTINE SIM( X, Y, I4, JIM, II )
C
C     THIS SUBROUTINE PLOTS THE TIME SERIES OF HARMONIC OSCILLATIONS.
C
      COMMON PN, DT, AJJJ, IPLOT, IO
      DIMENSION X(1), Y(1), AA(4), BB(4)
      IF (JIM) 7, 7, 8
    7 IPLOT = IPLOT + 1
      PLOTI = IPLOT
      APP = AJJJ + PLOTI
      IF (APP.GT.1.99999) CALL PLOT(T, 0.0, -3)
C     CALL SYMBOL( 0.0, 10.50, .11, 7HFIG.   ., 0.0, 7 )
C     CALL NUMBER( .55, 10.50, .11, AJJJ,    0.0, -1 )
C     CALL NUMBER( .77, 10.50, .11, PLOTI,   0.0, -1 )
      IF (IO.EQ.1) GO TO 2
      IF(IPLOT.EQ.4) CALL SYMBOL(-0.5, 4.5, 0.22, 7HSM.     , 90.0, 7)
    2 CONTINUE
      XMIN = 0.0
      DX = 25.
      AA(1) = 0.0
      AA(2) = PN
      AA(3) = PN
      AA(4) = 0.0
      YMAX = 0.15
      YMIN =-0.15
      IF (I4) 1, 3, 1
C
C     TO SCALE Y-AXIS.
C
    1 CALL SCALING( Y, II, YMAX, YMIN, IPLOT )
    3 DY = (YMAX-YMIN)/10.0
      WRITE (6,331) YMIN, YMAX, DY, IPLOT
  331 FORMAT(//,3X,*YMIN=*,F10.5,*YMAX=*,F10.5,10X,*DY=*,F10.5,*IPLOT=*,
     1I4,//)
      BB(1) = YMAX
      BB(2) = YMAX
      BB(3) = YMIN
      BB(4) = YMIN
      CALL LINE( AA, BB, 4, 1, 0, 3, XMIN, DX, YMIN, DY, 0.08 )
      CALL SYMBOL( 3.0, 0.28,0.25, 10HTIME(SEC.), 0., 10 )
      DO 12 I = 1, 5
      T = FLOAT(2*I) - 1.0
   12 CALL SYMBOL( 0., T, 0.1, 13, 90.0, -1 )
      IN = PN/DX
      DO 10 I = 1, IN
      T = FLOAT(I)
   10 CALL SYMBOL( T, 0., 0.1, 13, 0.0, -1 )
      IN = INT( PN/DX*0.5 ) + 1
      DO 11 I = 1, IN
      T = FLOAT(2*I-2) - 0.1
      FPN = 2.*DX*(FLOAT(I)-1.0)
   11 CALL NUMBER( T, 0.35, 0.19, FPN, 0.0, -1 )
      CALL AXIS(0., 0., 1H , 1, 10., 90., YMIN, DY, 5.)
      T = PN/DX + 2.0
      ISYM = 3
      XMAX = PN/DX
      CALL PLOT( T.0, 5.0, 3 )
```

```
      CALL PLOT( XMAX, 5.0, 2 )                                          HARMPLT    59
      GO TO 91                                                           HARMPLT    60
    8 ISYMB = 11                                                         HARMPLT    61
   91 CALL LINE(X, Y, II, 1, JIM, ISYMB, XMIN, DX, YMIN, DY, 0.10 )      HARMPLT    62
      PRINT 4, TI                                                        HARMPLT    63
    4 FORMAT(* TI=*, I5)                                                 HARMPLT    64
      RETURN                                                             HARMPLT    65
      END                                                                HARMPLT    66
```

```
      SUBROUTINE SCALING( Y, II, YMAX, YMIN, IPLOT )                    SCALING     2
C                                                                       SCALING     3
C     SCALING Y-AXIS OF PLOTS                                           SCALING     4
C     Y-AXIS IS KEPT THE SAME FOR EVERY PLOT                            SCALING     5
C                                                                       SCALING     6
      DIMENSION Y(II)                                                   SCALING     7
      YMIN = 1.E+30                                                     SCALING     8
      YMAX =-1.E+30                                                     SCALING     9
      DO 1 I = 1, II                                                    SCALING    10
      IF (YMAX.LT.Y(I) ) YMAX=Y(I)                                      SCALING    11
    1 IF (YMIN.GT.Y(I)) YMIN = Y(I)                                     SCALING    12
      T = YMAX-YMIN                                                     SCALING    13
      YMIN = YMIN-0.1*T                                                 SCALING    14
      IF (T.LT.0.001) YMIN = AINT((YMIN-0.0005)*2000.0)/2000.           SCALING    15
      IF (T.LT.0.1. AND. T.GT.0.001) YMIN=AINT((YMIN-0.005)*200.)/200.  SCALING    16
      IF (T.GE.0.01) YMIN = AINT((YMIN-0.05)*20.)/20.0                  SCALING    17
      YMAX = YMAX + 0.1*T                                               SCALING    18
      IF (T.LT.0.01) YMAX = AINT((YMAX+ 0.005)*200.0)/200.0             SCALING    19
      IF (T.GE.0.01) YMAX = AINT((YMAX+0.05)*20.)/20.0                  SCALING    20
      IF (IPLOT.EQ.1) GO TO 330                                         SCALING    21
      YMM = YMIN/ (YMAX+1.E-30)                                         SCALING    22
      IF (YMM.GT.0.0) GO TO 330                                         SCALING    23
      T = -YMIN                                                         SCALING    24
      IF (YMM.GE.-1.0) T = YMAX                                         SCALING    25
      IF (T.GT.25.0) T = AINT(T/25.0)*25.0+25.0                         SCALING    26
      IF (T.GT.5.0. AND. T.LT.25.0) T=AINT(T/5.0)*5.0+5.0               SCALING    27
      IF (T.LE.5.0. AND. T.GT.1.0) T=AINT(T)+1.0                        SCALING    28
      IF (T.GE.0.25. AND. T.LE.1.0) T=AINT(T/0.25)*0.25+0.25            SCALING    29
      IF (T.LT.0.25.AND.T.GT.0.1) T=0.25                                SCALING    30
      IF (T.LE.0.1.AND. T.GE.0.05) T=0.1                                SCALING    31
      IF (T.LT.0.05) T = 0.05                                           SCALING    32
      YMAX = T                                                          SCALING    33
      YMIN =-T                                                          SCALING    34
  330 RETURN                                                            SCALING    35
      END                                                               SCALING    36
```

Sample for Program Synharm follows.

INPUT CARDS :-

```
1
   .31415   45.00000 0
  [illegible]   [illegible]   .35150   [illegible]   -.01422   -.02844   .01198
 [illegible]   [illegible]   -0.00000   [illegible]   -0.00000   -0.00000   -0.00000
1 1 0
```

OUT-OF-PLANE OSCILLATIONS OF SATELLITE 1975..... INPUT INFORMATION   DATA SET 1

R = 65.00000 FT.
OMEGA0 = .31416 RAD./SEC. = .05000 CPS.
OMEGA(1)= 0.00000 RAD./SEC. = 0.00000 CPS.
OMEGA(2)= 0.00000 RAD./SEC. = 0.00000 CPS.
OMEGA(3)= 0.00000 RAD./SEC. = 0.00000 CPS.
OMEGA(4)= .61975 RAD./SEC. = .09860 CPS.
OMEGA(5)= .61955 RAD./SEC. = .09859 CPS.
OMEGA(6)= .32[illegible]2 RAD./SEC. = .05155 CPS.
OMEGA(7)= .326[illegible]3 RAD./SEC. = .05221 CPS.

B MATRIX .....

0.00000	0.00000	0.00000	.02076	0.00000	-.00768	-.00774
0.00000	0.00000	0.00000	0.00000	.02076	.00768	-.00774
0.00000	0.00000	0.00000	.02076	0.00000	.00768	.00774
0.00000	0.00000	0.00000	0.00000	.02076	-.00768	.00774
0.00000	0.00000	.01872	-.01422	0.00000	0.00000	0.00000
0.00000	.01872	0.00000	0.00000	-.01422	0.00000	0.00000
.06477	0.00000	0.00000	0.00000	0.00000	0.00000	.00593

INITIAL Q MATRIX .....

-.03108
.03408
.05179
.04176
-.03472
-.03246
.01113

INITIAL V MATRIX .....

-0.00000
-0.00000
-0.00000
-0.00000
-0.00000
-.00374
-.00009

OUTPUTS OF SIMULATION -----

T(SEC)	PSI(1)	PSI(2)	PSI(3)	PSI(4)	THETA1	THETA2	Z
0	[illegible]	.04149	.05167	.04264	-.01422	-.02866	.01199
1	-.01231	.03369	[illegible]	.03391	-.01158	-.02315	.01135
2	-.01776	.01345	[illegible]	.01358	-.00463	-.00926	.00951
3	-.02315	-.01177	.01136	-.01191	.00404	.00808	.00657
4	[illegible]	-.03263	[illegible]	-.03283	.01121	.02242	.00313
5	-.01393	-.04129	[illegible]	-.04168	.01421	.02842	-.00076
6	[illegible]	-.03565	[illegible]	-.03530	.01193	.02386	-.00456
7	[illegible]	[illegible]	[illegible]	-.01552	.00522	.01044	-.00753
8	[illegible]	[illegible]	-.03147	.00976	-.00343	-.00687	-.01033
9	[illegible]	.03176	[illegible]	.03137	-.01091	-.02162	-.01172
10	[illegible]	.04143	[illegible]	[illegible]	-.01417	-.02533	[illegible]
11	[illegible]	.03571	[illegible]	.03597	-.01226	-.02451	-.01075
12	[illegible]	[illegible]	[illegible]	.01717	-.00579	-.01158	[illegible]
13	.01316	[illegible]	-.01961	-.03742	.00283	.00566	[illegible]
14	[illegible]	-.03302	[illegible]	-.02979	.01040	.02390	-.00152
15	-.02577	[illegible]	-.01830	-.04051	.01410	.02420	[illegible]
16	[illegible]	-.03733	[illegible]	-.03531	.01256	.02513	[illegible]
17	-.03171	[illegible]	.01726	-.01797	.00636	.01271	[illegible]
18	[illegible]	[illegible]	[illegible]	.03549	[illegible]	-.00442	[illegible]
19	[illegible]	.02194	[illegible]	.03324	-.03596	-.01992	.01135
20	[illegible]	.04155	[illegible]	.04391	-.01403	-.02803	.01153
21	[illegible]	[illegible]	[illegible]	.03712	-.01266	-.02560	.01033
22	[illegible]	.02141	[illegible]	.01951	-.00693	-.01381	.00793
23	-.01767	-.00174	[illegible]	-.00555	.00159	.00320	.00395
24	[illegible]	-.02572	[illegible]	-.02070	[illegible]	.01902	.00016
25	[illegible]	-.03365	[illegible]	-.04150	.01368	.02777	-.00371
26	-.01155	-.03724	[illegible]	-.03323	.01310	.02620	-.00715
27	[illegible]	[illegible]	[illegible]	-.02253	.00745	.01489	-.00895
28	[illegible]	.00136	[illegible]	.00236	-.00097	-.00195	[illegible]
29	[illegible]	.02549	-.01767	.02627	-.00933	-.01906	[illegible]
30	[illegible]	.03177	[illegible]	.04041	-.01373	-.02746	-.01111
31	[illegible]	.03717	[illegible]	.03964	-.01333	-.02665	-.00913
32	[illegible]	.02214	[illegible]	.02436	-.00797	-.01593	-.00612
33	[illegible]	-.00229	[illegible]	.00033	.00036	.00071	-.00253
34	-.01455	-.02543	[illegible]	-.02145	.00855	.01709	.00133
35	-.03159	-.04126	[illegible]	-.03912	.01356	.02712	.00514
36	[illegible]	-.04177	[illegible]	-.03924	.01353	.02707	.00636
37	[illegible]	-.02365	.01625	-.02393	.00848	.01696	.01056
38	[illegible]	-.00124	[illegible]	-.00074	.00027	.00054	.01145
39	[illegible]	.02135	[illegible]	.02337	-.00803	-.01607	.01173
40	[illegible]	.03364	[illegible]	.03874	-.01335	-.02671	.01049
41	[illegible]	[illegible]	[illegible]	.03446	-.01371	-.02742	.00837
42	[illegible]	.02777	[illegible]	.02460	-.00897	-.01793	.00493
43	[illegible]	.00445	[illegible]	.00076	-.00069	-.00178	.00122
44	[illegible]	-.02172	[illegible]	-.02395	.00752	.01504	-.00295
45	[illegible]	-.03455	[illegible]	-.04011	.01313	.02626	-.00643
46	[illegible]	[illegible]	[illegible]	-.04192	.01387	.02773	[illegible]
47	[illegible]	[illegible]	[illegible]	-.02851	.00945	.01889	-.01124
48	[illegible]	-.00411	[illegible]	-.00473	.00152	.00303	-.01135
49	[illegible]	.01939	[illegible]	.02076	-.00699	-.01395	-.01143
50	[illegible]	.03353	[illegible]	.03966	-.01288	-.02575	-.00954
51	[illegible]	[illegible]	[illegible]	.04250	-.01399	-.02797	-.00645
52	[illegible]	.02341	[illegible]	.03133	-.00993	-.01999	-.00335
53	[illegible]	[illegible]	[illegible]	.03955	-.00213	[illegible]	[illegible]

SEC.	THETA1	THETA2	XCONIC	YCONIC
0.0000	-.01423	-.02844	-.01422	-.02845
.0625	-.01421	-.02842	-.01365	-.02870
.1250	-.01419	-.02835	-.01305	-.02890
.1875	-.01412	-.02825	-.01244	-.02904
.2500	-.01403	-.02810	-.01180	-.02912
.3125	-.01395	-.02791	-.01115	-.02915
.3750	-.01384	-.02768	-.01049	-.02912
.4375	-.01373	-.02740	-.00982	-.02903
.5000	-.01354	-.02709	-.00914	-.02889
.5625	-.01337	-.02673	-.00846	-.02867
.6250	-.01317	-.02633	-.00778	-.02840
.6875	-.01293	-.02590	-.00710	-.02809
.7500	-.01271	-.02542	-.00643	-.02769
.8125	-.01245	-.02491	-.00576	-.02725
.8750	-.01219	-.02436	-.00511	-.02676
.9375	-.01183	-.02378	-.00447	-.02621
1.0000	-.01151	-.02315	-.00385	-.02560
1.0625	-.01125	-.02249	-.00326	-.02494
1.1250	-.01091	-.02181	-.00268	-.02424
1.1875	-.01054	-.02109	-.00213	-.02348
1.2500	-.01015	-.02033	-.00161	-.02267
1.3125	-.00977	-.01954	-.00112	-.02182
1.3750	-.00935	-.01873	-.00066	-.02093
1.4375	-.00894	-.01789	-.00024	-.02000
1.5000	-.00851	-.01702	.00015	-.01903
1.5625	-.00806	-.01612	.00049	-.01802
1.6250	-.00761	-.01520	.00080	-.01699
1.6875	-.00713	-.01426	.00106	-.01591
1.7500	-.00663	-.01330	.00129	-.01481
1.8125	-.00615	-.01231	.00145	-.01369
1.8750	-.00565	-.01131	.00158	-.01255
1.9375	-.00513	-.01029	.00166	-.01139
2.0000	-.00463	-.00925	.00170	-.01021
2.0625	-.00411	-.00821	.00168	-.00902
2.1250	-.00357	-.00715	.00162	-.00783
2.1875	-.00304	-.00609	.00151	-.00663
2.2500	-.00251	-.00503	.00135	-.00542
2.3125	-.00195	-.00391	.00114	-.00422
2.3750	-.00141	-.00282	.00088	-.00302
2.4375	-.00085	-.00172	.00057	-.00183
2.5000	-.00031	-.00062	.00022	-.00066
2.5625	.00024	.00049	-.00018	.00051
2.6250	.00079	.00159	-.00063	.00166
2.6875	.00134	.00268	-.00111	.00279
2.7500	.00189	.00379	-.00165	.00389
2.8125	.00243	.00485	-.00222	.00497
2.8750	.00297	.00595	-.00283	.00602
2.9375	.00351	.00702	-.00349	.00703
3.0000	.00404	.00809	-.00416	.00802
3.0625	.00455	.00913	-.00489	.00897
3.1250	.00501	.01017	-.00563	.00987
3.1875	.00553	.01119	-.00641	.01074
3.2500	.00603	.01219	-.00721	.01157
3.3125	.00651	.01313	-.00804	.01235
3.3750	.00707	.01414	-.00889	.01308
3.4375	.00754	.01509	-.00975	.01377
3.5000	.00801	.01601	-.01063	.01440
3.5625	.00843	.01691	-.01153	.01499
3.6250	.00883	.01778	-.01243	.01552

Appendix A Conservation of X and Y Momenta

The equations yielded by using Lagrangian method are those corresponding to the motions that nature chooses, according to the philosophy of Lagrangian Principle of Least Action. The motions so derived should be consistent with conservation laws of nature. Therefore, the X and Y -equations of motion should be derivable by considering X and Y linear momenta conservation. This approach provides some checking on possible algebraic mistakes in the calculations using the other method.

In the center of mass system, the linear momentum of the whole system in X - direction should be zero. The total X - momentum comprises of the following :

$$X \text{ - momentum of the hub} = M\dot{X}$$

$$X \text{ - momentum of the tip masses} = \sum_{i=1}^{4} m_i(\dot{X} + \dot{x}_i(t) - \dot{\theta}(t)\, y_i(t))$$

$$X \text{ - momentum of the wires} = \sum_{i=1}^{4} \int dr_i\, \rho\, (\dot{X} + \dot{x}_i(t) - \dot{\theta}(t)\, y_i(t))$$

Changing to polar coordinates (see page 3, 4,) we have:

$$\dot{x}_i(t) - \dot{\theta}(t) y_i(t) = \dot{r}_i \cos\phi_i - r_i \sin\phi_i\, (\dot{\theta} + \dot{\phi}_i) - r_o \sin\Theta_i\, \dot{\theta}$$

Thus, the X - momentum of the entire system is found as:

$$M\dot{X} + \sum_{i=1}^{4} \left[(m + \rho r_i)(\dot{X} + \dot{r}_i \cos\phi_i - r_o \dot{\theta} \sin\Theta_i) - \left(m r_i + \rho \frac{r_i^2}{2} \right) \right.$$

$$\left. (\dot{\theta} + \dot{\phi}_i) \sin\phi_i \right] = 0 \text{ (equates to zero)} \qquad (A-1)$$

Similarly, the Y -momentum of the entire system is found as:

$$M\dot{Y} + \sum_{i=1}^{4} \left[(m + \rho r_i)(\dot{Y} + \dot{r}_i \sin\phi_i + r_o \cos\Theta_i \dot{\theta}) + \left(m r_i + \rho \frac{r_i^2}{2} \right) \right.$$

$$(\dot{\theta} + \dot{\phi}_i) \cos \phi_i] = 0 \text{ (equates to zero)} \qquad \text{(A-2)}$$

The time derivatives of the linear momenta equations must yield the corresponding equations of motion. Thus, differenting the X - momenta equation, we have the following X -equation of Motion:

$$M\ddot{X} + \sum_{i=1}^{4} \Big\{ \rho r_i (\dot{X} + \dot{r}_i \cos\Phi_i - r_o \sin\Theta_i \dot{\theta}) + (m + \rho r_i)$$

$$(\ddot{X} - 2 \dot{r}_i \sin \Phi_i [\dot{\theta} + \dot{\phi}_i] - r_o \sin\Theta_i \ddot{\theta} - r_o \cos\Theta_i \dot{\theta}^2) \quad -(mr_i + \rho \frac{r_i^2}{2})$$

$$([\ddot{\theta} + \ddot{\phi}_i] \sin\Phi_i + [\dot{\theta} + \dot{\phi}_i]^2 \cos \Phi_i) \Big\} = 0 \qquad \text{(A-3)}$$

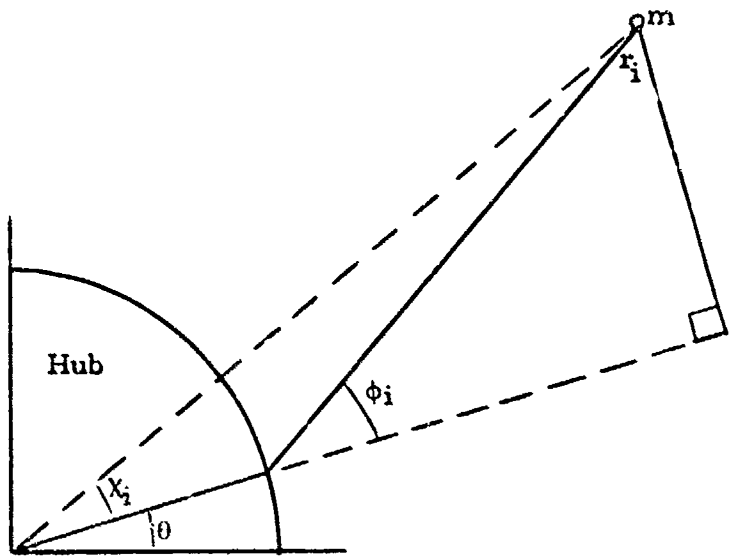
Similarly, the Y -equation of Motion is obtained as follows:

$$M\ddot{Y} + \sum_{i=1}^{4} \Big\{ \rho r_i (\dot{Y} + \dot{r}_i \sin \Phi_i + r_o \dot{\theta} \cos\Theta_i) + (m + \rho r_i)$$

$$(\ddot{Y} + 2 \dot{r}_i \cos\Phi_i [\dot{\theta} + \dot{\phi}_i] - r_o \sin\Theta_i \dot{\theta}^2 + r_o \cos\Theta_i \ddot{\theta}) \; -$$

$$(mr_i + \rho \frac{r_i^2}{2}) ([\dot{\theta} + \dot{\phi}_i]^2 \sin \Phi_i - [\ddot{\theta} + \ddot{\phi}_i] \cos \Phi_i) \Big\} = 0 \qquad \text{(A-4)}$$

Appendix B Conservation of Angular Momentum



For simplicity, it is sufficient to study the case of one boom; (for 4 booms, let $r \rightarrow r_i$ and $\phi \rightarrow \phi_i$ with summation over $i = 1, \ldots 4$). Let $\rho \rightarrow 0$ for further simplicity; (integration over r if $\rho \neq 0$).

Total moment of inertia I_T is the sum of that of the hub plus that of the boom (s):

$$I_T = I_o + m (r^2 + r_o^2 + 2rr_o \cos \phi) \tag{B-1}$$

The total angular momentum about the center of the hub is the sum of that of the hub plus that of the boom(s):

$$I_o \dot{\theta} + m (r^2 + r_o^2 + 2rr_o \cos\phi) (\dot{\theta} + \dot{\chi})$$

$$= I_T \dot{\theta} + m (r^2 + r_o + 2rr_o \cos \phi) \dot{\chi} = I_T \dot{\theta} + I_m \dot{\chi}$$

where $\chi = \tan^{-1} \left(\frac{r \sin \phi}{r_o + r \cos\phi} \right)$

and $\dot{\chi} = \frac{d}{dt} \tan^{-1} \left(\frac{r \sin \phi}{r_o + r \cos\phi} \right)$

Since $\frac{d}{dt}\tan^{-1}(x) = \frac{1}{1+x^2}\,\dot{x}$, we have:

$$\dot{\chi} = \frac{1}{1 + \dfrac{r^2\sin^2\phi}{r_o^2 + 2r_o r\cos\phi + r^2\cos^2\phi}}\left[\frac{\dot{r}\sin\phi + r\cos\phi\,\dot{\phi}}{r_o + r\cos\phi} + \right.$$

$$\left. - \frac{r\sin\phi}{(r_o + r\cos\phi)^2}\,(\dot{r}\cos\phi - r\sin\phi\,\dot{\phi})\right]$$

$$= \frac{(r_o + r\cos\phi)^2}{r_o^2 + 2r_o r\cos + r^2}\left[\frac{(r_o + r\cos\phi)(\dot{r}\sin\phi + r\cos\phi\,\dot{\phi})}{(r_o + r\cos\phi)^2}\right.$$

$$\left. + \frac{r\sin\phi\,(r\sin\phi\,\dot{\phi} - \dot{r}\cos\phi)}{(r_o + r\cos\phi)^2}\right]$$

$$= \frac{r_o(\dot{r}\sin\phi + r\cos\phi\,\dot{\phi}) + r^2\dot{\phi}}{r^2 + 2rr_o\cos\phi + r_o^2}$$

Hence, the total angular momentum becomes

$$I_T\,\dot{\theta} + I_m\,\dot{\chi} = I_T\,\dot{\theta} + m\left[r_o(\dot{r}\sin\phi + r\cos\phi\,\dot{\phi}) + r^2\dot{\phi}\right] \qquad (B-2)$$

If there is no external damping or force, the total angular momentum of the entire system should be a constant of motion. Hence,

$$\frac{d}{dt}(I_T\,\dot{\theta} + I_m\,\dot{\chi}) = 0 \qquad (B-3)$$

This conservation law should yield a θ-equation of motion identical to that obtained by using Lagrangian method.

$$\frac{d}{dt}(I_T\,\dot{\theta} + I_m\,\dot{\chi}) = \dot{I}_T\,\dot{\theta} + I_T\,\ddot{\theta} + m\,\frac{d}{dt}\left[r_o(\dot{r}\sin\phi + r\cos\phi\,\dot{\phi}) + r^2\dot{\phi}\right]$$

$$+ \frac{d}{dt}\left[I_o + m(r^2 + r_o^2 + 2rr_o\cos\phi)\right]\dot{\theta} + I_T\,\ddot{\theta}$$

$$= m\left[r_o(2\dot{r}\cos\phi\,\dot{\phi} + r\cos\phi\,\ddot{\phi} - r\sin\phi\,\dot{\phi}^2) + 2r\dot{r}\dot{\phi} + r^2\ddot{\phi}\right]$$

$$= I_T\ddot{\theta} + m(2r\dot{r} + 2\dot{r}r_o\cos\phi - 2rr_o\sin\phi\dot{\phi})\dot{\theta} +$$

$$m\{r_o(2\dot{r}\cos\phi\dot{\phi} + r\cos\phi\ddot{\phi} - r\sin\phi\dot{\phi}^2) + 2r\dot{r}\dot{\phi} + r^2\ddot{\phi}\}$$

$$= I_T\ddot{\theta} + m\left\{\ddot{\phi}(r^2 + rr_o\cos\phi) + 2(\dot{\theta}+\dot{\phi})(r\dot{r} + r_o\dot{r}\cos\phi - r_o r\sin\phi\dot{\phi})\right.$$

$$\left.+ r r_o\sin\phi\dot{\phi}_i^2\right\} = 0 \quad \text{(equate to zero)}$$

which is the desired θ-equation for one boom.

For 4 booms with $\rho\neq 0$, the total angular momentum of the system becomes

$$I_T\dot{\theta} + \sum_{i=1}^{4}\left\{A_i\dot{\phi}_i + B_i r_o\cos\phi_i\dot{\phi}_i + D_i r_o\dot{r}_i\sin\phi_i\right\} = \text{constant of motion}$$

Differentiating the angular momentum should give the θ-equation of motion:

$$I_T\ddot{\theta} + \dot{I}_T\dot{\theta} + \sum_{i=1}^{4}\left\{2B_i\dot{r}_i\dot{\phi}_i + D_i\dot{r}_i r_o\cos\phi_i\dot{\theta} + A_i\ddot{\phi}_i + \right.$$

$$\left. B_i r_o\cos\phi_i\ddot{\phi}_i - B_i r_o\sin\phi_i\dot{\phi}_i^2 + D_i r_o\dot{r}_i\cos\phi_i\dot{\phi}_i + \rho\dot{r}_i r_o\dot{r}_i\sin\phi_i\right\}$$

$$= I_T\ddot{\theta} + \sum_{i=1}^{4}\left\{2B_i\dot{r}_i + 2D_i\dot{r}_i r_o\cos\phi_i + \rho\dot{r}_i r_o^2 - 2B_i r_o\sin\phi_i\dot{\phi}_i\right\}\dot{\theta}$$

$$+ \sum_{i=1}^{4}\left\{\ddot{\phi}_i(A_i + B_i r_o\cos\phi_i) + 2(B_i\dot{r}_i + D_i r_o\dot{r}_i\cos\phi_i - \right.$$

$$\left. B_i r_o\sin\phi_i\dot{\phi}_i)\dot{\phi}_i + B_i r_o\sin\phi_i\dot{\phi}_i^2 + \rho\dot{r}_i^2 r_o\sin\phi_i\right\}$$

$$= I_T\ddot{\theta} + \sum_{i=1}^{4}\left\{2(\dot{\theta}+\dot{\phi}_i)(B_i\dot{r}_i + D_i r_o\dot{r}_i\cos\phi_i - B_i r_o\sin\phi_i\dot{\phi}_i)\right.$$

$$\left. + \ddot{\phi}_i(A_i + B_i r_o\cos\phi_i) + B_i r_o\sin\phi_i\dot{\phi}_i^2 + \rho\dot{r}_i^2 r_o\sin\phi_i + \rho\dot{r}_i r_o^2\right\}$$

$= 0$ (equate to zero)

which is the same as that derived by using the Lagrangian method. It is the θ - equation of motion for pure rotation without translation. (The symbols A, B, etc are defined in Chapter 3).

Appendix C To Derive Some Simple Frequencies

Consider a simple situation with the following assumptions:

1. No wire mass density
2. No boom deployment/retraction
3. $\dot{\phi}$ and ϕ are small, but $\dot{\theta}$ is not small
4. $\ddot{\theta}$, $\ddot{\phi}$ are not negligible
5. No translation and no damping
6. Only one boom, all booms in phase, or all out of phase

Such simple case gives:

$$\phi\text{-equ:} \qquad mr^2 \ddot{\phi} + \ddot{\theta} (mr^2 + mrr_o) + \dot{\theta}^2 mrr_o \sin\phi = 0$$

$$\text{or:} \qquad r^2 \ddot{\phi} + \ddot{\theta} (r^2 + rr_o) + \dot{\theta}^2 rr_o \phi = 0 \qquad (C\text{-}1)$$

$$\theta\text{-equ:} \qquad I_T \ddot{\theta} + mr(r + r_o)\ddot{\phi} = 0 \qquad (C\text{-}2)$$

Note θ-equ. can be derived from Lagrangian L of the total system, or from angular momentum:

$$\frac{d}{dt}\left[I_o \dot{\theta} + m(r^2 + r_o^2 + 2rr_o \cos\phi)(\dot{\theta} + \dot{\phi})\right]$$

$$\approx \frac{d}{dt}\left[I_T \dot{\theta} + m(r^2 + r_o^2 + 2rr_o)(\dot{\phi})\right] \approx I_T \ddot{\theta} + mr(r + r_o)\ddot{\phi}$$

≈ 0 (equate to zero because it is constant of motion)

where I_T is assumed constant because $\dot{\phi}$ and ϕ are negligible so that

$$I_T \approx I_o + m(r^2 + r_o^2 + 2r_o r)$$

Uncoupled Frequency

If the hub is uncoupled to the vibration of boom, then the hub rotates with constant $\dot{\theta}$ ($\equiv \omega$), so that equ. (C-1) becomes:

$$r^2 \ddot{\phi} + \dot{\theta}^2 \; r \, r_o \, \phi = 0$$

$$\text{or} \quad \ddot{\phi} + \omega^2 \frac{r_o}{r} \phi = 0$$

Hence the uncoupled frequency $\quad \Omega = \omega \sqrt{\dfrac{r_o}{r}}$ (C-3)

Coupled Frequency

In the coupled case, $\ddot{\theta}$ is dependent on $\ddot{\phi}$. Combining equs (C-1) and (C-2), we have:

$$r^2 \ddot{\phi} + \left[- \frac{mr\,(r + r_o)}{I_T} \ddot{\phi} \right] (r^2 + r\,r_o) + \dot{\theta}^2 \, r\,r_o \, \phi = 0$$

$$\left[r^2 - \frac{r\,(r + r_o\,)\,m}{I_o + m\,(r + r_o)^2} (r^2 + r\,r_o\,) \right] \ddot{\phi} + \dot{\theta}^2 \, r\,r_o\, o = 0$$

$$\left[\frac{\left[I_o + m\,(r + r_o)^2 \right] r^2 - r^2\,(r + r_o\,)^2\, m}{I_T} \right] \ddot{\phi} + \dot{\theta}^2 \; r\,r_o\,\phi = 0$$

$$\frac{I_o\, r^2}{I_T} \ddot{\phi} + \dot{\theta}^2 \, r\,r_o\,\phi = 0$$

$$\ddot{\phi} + \dot{\theta}^2 \frac{I_T}{I_o} \frac{r_o}{r} \phi = 0$$

Hence the coupled frequency $\quad \Omega = \omega \sqrt{\dfrac{I_T}{I_o} \; \dfrac{r_o}{r}}$ (C-4)

Appendix D Characteristic Determinant for Inplane Dynamics

In this appendix, the following symbols are used:

$$
\text{Let} \quad \begin{aligned}
a &\equiv a\omega^2 - p \\
b &\equiv b\omega^2 \\
M &\equiv \mathcal{M}\omega^2 \\
d &\equiv d\omega^2 \\
e &\equiv e\omega^2 \\
c &\equiv c\omega^2
\end{aligned}
$$

The right hand sides of the above identifications have the same symbols used in the chapters, while LHS symbols are specifically for this appendix only.

$$
\det|A| = \begin{vmatrix}
a & 0 & 0 & 0 & b & -d & e \\
0 & a & 0 & 0 & b & -e & -d \\
0 & 0 & a & 0 & b & d & -e \\
0 & 0 & 0 & a & b & e & d \\
b & b & b & b & c & 0 & 0 \\
-d & -e & d & e & 0 & M & 0 \\
e & -d & -e & d & 0 & 0 & M
\end{vmatrix}
\xrightarrow[R_7 - \frac{de}{d^2+e^2} R_6]{R_6 + \frac{d}{e} R_7}
$$

$$
\begin{vmatrix}
a & 0 & 0 & 0 & b & -d & e \\
0 & a & 0 & 0 & b & -e & -d \\
0 & 0 & a & 0 & b & d & -e \\
0 & 0 & 0 & a & b & e & d \\
b & b & b & b & c & 0 & 0 \\
0 & \frac{d^2+e^2}{e} & 0 & \frac{d^2+e^2}{e} & 0 & M & \frac{d}{e}M \\
e & 0 & -e & 0 & 0 & -\frac{deM}{d^2+e^2} & \frac{e^2M}{d^2+e^2}
\end{vmatrix}
\xrightarrow[C_6 - \frac{ed}{e^2+d^2} C_7]{C_7 + \frac{e}{d} C_6}
\begin{vmatrix}
a & 0 & 0 & 0 & b & -d & 0 \\
0 & a & 0 & 0 & b & 0 & -\frac{d^2+e^2}{d} \\
0 & 0 & a & 0 & b & d & 0 \\
0 & 0 & 0 & a & b & 0 & \frac{d^2+e^2}{d} \\
b & b & b & b & c & 0 & 0 \\
0 & -\frac{d^2+e^2}{e} & 0 & \frac{d^2+e^2}{e} & 0 & 0 & \frac{d^2+e^2}{de}M \\
e & 0 & -e & 0 & 0 & -\frac{de}{d^2+e^2}M & 0
\end{vmatrix}
$$

$$
\xrightarrow[]{\frac{C_6}{d}\,,\ dC_7}
\begin{vmatrix}
a & 0 & 0 & 0 & b & -1 & 0 \\
0 & a & 0 & 0 & b & 0 & -(d^2+e^2) \\
0 & 0 & a & 0 & b & 1 & 0 \\
0 & 0 & 0 & a & b & 0 & (d^2+e^2) \\
b & b & b & b & c & 0 & 0 \\
0 & -(b^2+e^2) & 0 & d^2+e^2 & 0 & 0 & (d^2+e^2)M \\
1 & 0 & -1 & 0 & 0 & \frac{-M}{d^2+e^2} & 0
\end{vmatrix}
$$

$$
\xrightarrow[R_6/(d^2+e^2)]{C_7/(d^2+e^2)}
\begin{vmatrix}
a & 0 & 0 & 0 & b & -1 & 0 \\
0 & a & 0 & 0 & b & 0 & -1 \\
0 & 0 & a & 0 & b & 1 & 0 \\
0 & 0 & 0 & a & b & 0 & 1 \\
b & b & b & b & c & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & M/(d^2+e^2) \\
1 & 0 & -1 & 0 & 0 & -M/(d^2+e^2) & 0
\end{vmatrix}
$$

For further simplification of notation, let us denote, in the rest of this appendix, the following:

$$M \equiv M\ /\ (d^2+e^2)$$

$$
D \equiv \begin{vmatrix}
a & 0 & 0 & 0 & b \\
0 & a & 0 & 0 & b \\
0 & 0 & a & 0 & b \\
0 & 0 & 0 & a & b \\
b & b & b & b & c
\end{vmatrix}
$$

Then, the determinant $|A|$ is expanded into a sum of terms by means of Laplace's expansion:

$$
\begin{vmatrix}
a & & & & b & -1 & 0 \\
 & a & & & b & 0 & -1 \\
 & & a & & b & 1 & 0 \\
 & & & a & b & 0 & 1 \\
b & b & b & b & c & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & M & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & M
\end{vmatrix}
$$

$$
= \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix}
\begin{vmatrix}
0 & 0 & a & 0 & b \\
0 & 0 & 0 & a & b \\
b & b & b & b & c \\
-1 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 1 & 0
\end{vmatrix}
+ \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}
\begin{vmatrix}
0 & a & 0 & 0 & b \\
0 & 0 & a & 0 & b \\
b & b & b & b & c \\
-1 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 1 & 0
\end{vmatrix}
$$

$$
- \begin{vmatrix} -1 & 0 \\ 0 & M \end{vmatrix}
\begin{vmatrix}
0 & a & 0 & 0 & b \\
0 & 0 & a & 0 & b \\
0 & 0 & 0 & a & b \\
b & b & b & b & c \\
-1 & 0 & 1 & 0 & 0
\end{vmatrix}
+ \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}
\begin{vmatrix}
a & 0 & 0 & 0 & b \\
0 & 0 & 0 & a & b \\
b & b & b & b & c \\
-1 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 1 & 0
\end{vmatrix}
$$

$$
- \begin{vmatrix} 0 & -1 \\ M & 0 \end{vmatrix}
\begin{vmatrix}
a & 0 & 0 & 0 & b \\
0 & 0 & a & 0 & b \\
0 & 0 & 0 & a & b \\
b & b & b & b & c \\
0 & -1 & 0 & 1 & 0
\end{vmatrix}
+ \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}
\begin{vmatrix}
a & 0 & 0 & 0 & b \\
0 & a & 0 & 0 & b \\
b & b & b & b & c \\
-1 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 1 & 0
\end{vmatrix}
$$

$$
- \begin{vmatrix} 1 & 0 \\ 0 & M \end{vmatrix}
\begin{vmatrix}
a & 0 & 0 & 0 & b \\
0 & a & 0 & 0 & b \\
0 & 0 & 0 & a & b \\
b & b & b & b & c \\
-1 & 0 & 1 & 0 & 0
\end{vmatrix}
- \begin{vmatrix} 0 & 1 \\ M & 0 \end{vmatrix}
\begin{vmatrix}
a & 0 & 0 & 0 & b \\
0 & a & 0 & 0 & b \\
0 & 0 & a & 0 & b \\
b & b & b & b & c \\
0 & -1 & 0 & 1 & 0
\end{vmatrix}
+ M^2 D
$$

$$
= \sum_{i=1}^{9} I_i
$$

$$I_1 = \begin{vmatrix} a & 0 \\ 0 & a \end{vmatrix} \begin{vmatrix} b & b & c \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{vmatrix} - \begin{vmatrix} a & b \\ 0 & b \end{vmatrix} \begin{vmatrix} b & b & b \\ -1 & 0 & 0 \\ 0 & -1 & 1 \end{vmatrix} + \begin{vmatrix} b & b & b \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix} \begin{vmatrix} 0 & b \\ a & b \end{vmatrix}$$

$$= a^2 (c) - ab (b+b) + (-ab) (b+b) = a^2c - 4ab^2$$

$$I_2 = \begin{vmatrix} a & 0 \\ 0 & a \end{vmatrix} \begin{vmatrix} b & b & c \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} - \begin{vmatrix} a & b \\ 0 & b \end{vmatrix} \begin{vmatrix} b & b & b \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & b \\ a & b \end{vmatrix} \begin{vmatrix} b & b & b \\ -1 & 0 & 0 \\ 0 & -1 & 1 \end{vmatrix}$$

$$= - a^2 (-c) - ab (b + b) - ab (b + b) = a^2c - 4ab^2$$

$$I_3 = M \begin{vmatrix} a & b \\ 0 & b \end{vmatrix} \begin{vmatrix} 0 & 0 & a \\ b & b & b \\ -1 & 0 & 0 \end{vmatrix} + M \begin{vmatrix} a & 0 \\ 0 & a \end{vmatrix} \begin{vmatrix} 0 & a & b \\ b & b & c \\ -1 & 0 & 0 \end{vmatrix} - M \begin{vmatrix} 0 & b \\ a & b \end{vmatrix} \begin{vmatrix} 0 & 0 & a \\ b & b & b \\ -1 & 0 & 0 \end{vmatrix}$$

$$= M\, ab\, (ab + ab) + M a^2 (-ac + b^2) + Mab\, (ab) = M (4 a^2 b^2 - a^3 c)$$

$$I_4 = \begin{vmatrix} a & 0 \\ 0 & a \end{vmatrix} \begin{vmatrix} b & b & c \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{vmatrix} - \begin{vmatrix} a & b \\ 0 & b \end{vmatrix} \begin{vmatrix} b & b & b \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & b \\ a & b \end{vmatrix} \begin{vmatrix} b & b & b \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix}$$

$$= a^2 (c) - ab (b + b) - ab (b + b) = a^2 c - 4 a b^2$$

$$I_5 = +M \begin{vmatrix} a & 0 \\ 0 & a \end{vmatrix} \begin{vmatrix} 0 & a & b \\ b & b & c \\ -1 & 1 & 0 \end{vmatrix} +M \begin{vmatrix} a & b \\ 0 & b \end{vmatrix} \begin{vmatrix} 0 & 0 & a \\ b & b & b \\ -1 & 0 & 1 \end{vmatrix} +M \begin{vmatrix} 0 & b \\ a & b \end{vmatrix} \begin{vmatrix} 0 & 0 & a \\ b & b & b \\ 0 & -1 & 1 \end{vmatrix}$$

$$= M a^2 (-ac + b^2 + b^2) + M\, ab\, (ab) - M a b (- ab) = -M (a^3 c - 4a^2b^2)$$

$$I_6 = \begin{vmatrix} a & 0 \\ 0 & a \end{vmatrix} \begin{vmatrix} b & b & c \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} - \begin{vmatrix} a & b \\ 0 & b \end{vmatrix} \begin{vmatrix} b & b & b \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & b \\ a & b \end{vmatrix} \begin{vmatrix} b & b & b \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= a^2 (c) - ab (b + b) - ab (b + b) = a^2 c - 4ab^2$$

$$I_7 = -M \begin{vmatrix} a & 0 \\ 0 & a \end{vmatrix} \begin{vmatrix} 0 & a & b \\ b & b & c \\ 1 & 0 & 0 \end{vmatrix} + M \begin{vmatrix} a & b \\ 0 & b \end{vmatrix} \begin{vmatrix} 0 & 0 & a \\ b & b & b \\ 0 & 1 & 0 \end{vmatrix} - M \begin{vmatrix} 0 & b \\ a & b \end{vmatrix} \begin{vmatrix} 0 & 0 & a \\ b & b & b \\ -1 & 1 & 0 \end{vmatrix}$$

$$= -M a^2 (ac - b^2) + M a b (ab) + M a b (ab + ab) = M (4 a^2 b^2 - a^3 c)$$

$$I_8 = M \begin{vmatrix} a & 0 \\ 0 & a \end{vmatrix} \begin{vmatrix} a & 0 & b \\ b & b & c \\ 0 & 1 & 0 \end{vmatrix} - M \begin{vmatrix} a & b \\ 0 & b \end{vmatrix} \begin{vmatrix} 0 & a & 0 \\ b & b & b \\ -1 & 0 & 1 \end{vmatrix} + M \begin{vmatrix} 0 & b \\ a & b \end{vmatrix} \begin{vmatrix} 0 & a & 0 \\ b & b & b \\ 0 & 0 & 1 \end{vmatrix}$$

$$= M a^2 (b^2 - ac) - M a b (- ab - ab) + M (-ab)(-ab) = M(4a^2b^2 - a^3 c)$$

$$I_9 = M^2 \begin{vmatrix} a & 0 \\ 0 & a \end{vmatrix} \begin{vmatrix} a & 0 & b \\ 0 & a & b \\ b & b & c \end{vmatrix} - M^2 \begin{vmatrix} a & b \\ 0 & b \end{vmatrix} \begin{vmatrix} 0 & a & 0 \\ 0 & 0 & a \\ b & b & b \end{vmatrix} + M^2 \begin{vmatrix} 0 & b \\ a & b \end{vmatrix} \begin{vmatrix} 0 & a & 0 \\ 0 & 0 & a \\ b & b & b \end{vmatrix}$$

$$= M^2 a^2 (a^2 c - ab^2 - b^2 a) - M^2 a b (a^2 b) + M^2 (-ab) (a^2 b)$$

$$= M^2 (a^4 c - 4a^3 b^2)$$

Collecting the results, we find,

$$\det |A| = \sum_{i=1}^{9} I_i = 4 (a^2 c - 4 ab^2) + 4M(4a^2 b^2 - a^3 c) + M^2 (a^4 c - 4 a^3 b^2)$$

$$= (Ma - 2)^2 a (ac - 4b^2)$$

Let us replace these symbols by those on the RHS of the identification equations on page 220. Thus:

$$\det |A| = \omega^6 \left[\mathcal{M} (a\omega^2 - p) - 2\omega^2 \left(mr + \rho \frac{r^2}{2}\right)^2 \right]^2 (a\omega^2 - p)$$

$$\left[(a\omega^2 - p)c - 4b^2\omega^2 \right] = 0$$

This is the characteristic or secular equation for in-plane normal mode oscillations.

Appendix E $\cos\Phi_i$ and $\sin\Phi_i$

$$\cos\Phi_1 = \cos\omega_o t\,[\,1 - \frac{(\theta' + \phi_1)^2}{2} + \ldots] - \sin\omega_o t\,[\,\theta' + \phi_1 + \ldots]$$

$$\cos\Phi_2 = -\sin\omega_o t\,[1 - \frac{(\theta' + \phi_2)^2}{2} + \ldots] - \cos\omega_o t\,[\,\theta' + \phi_2 \ldots]$$

$$\cos\Phi_3 = -\cos\omega_o t\,[\,1 - \frac{(\theta' + \phi_3)^2}{2} + \ldots] + \sin\omega_o t\,[\,\theta' + \phi_3 + \ldots]$$

$$\cos\Phi_4 = \sin\omega_o t\,[\,1 - \frac{(\theta' + \phi_4)^2}{2} + \ldots] + \cos\omega_o t\,[\,\theta' + \phi_4 + \ldots]$$

$$\begin{aligned}\sum_{i=1}^{4} \cos\Phi_i = {} & (\phi_3 - \phi_1)\sin\omega_o t + (\phi_4 - \phi_2)\cos\omega_o t\ + \\ & \frac{1}{2}\,[\,\phi_3^{\,2} - \phi_1^{\,2} + 2\theta'(\phi_3 - \phi_1)\,]\cos\omega_o t\ + \\ & \frac{1}{2}\,[\,\phi_2^{\,2} - \phi_4^{\,2} + 2\theta'(\phi_2 - \phi_4)\,]\sin\omega_o t + O(\phi_i^{\,3})\end{aligned}$$

$$\sin\Phi_1 = \sin\omega_o t\,[\,1 - \frac{(\theta' + \phi_1)^2}{2} + \ldots] + \cos\omega_o t\,[\,\theta' + \phi_1 + \ldots]$$

$$\sin\Phi_2 = \cos\omega_o t\,[\,1 - \frac{(\theta' + \phi_2)^2}{2} + \ldots] - \sin\omega_o t\,[\,\theta' + \phi_2 + \ldots]$$

$$\sin\Phi_3 = -\sin\omega_o t\,[\,1 - \frac{(\theta' + \phi_3)^2}{2} + \ldots] - \cos\omega_o t\,[\,\theta' + \phi_3 + \ldots]$$

$$\sin\Phi_4 = -\cos\omega_o t\,[\,1 - \frac{(\theta' + \phi_4)^2}{2} + \ldots] + \sin\omega_o t\,[\,\theta' + \phi_4 + \ldots]$$

$$\begin{aligned}\sum_{i=1}^{4} \sin\Phi_i = {} & (\phi_1 - \phi_3)\cos\omega_o t + (\phi_4 - \phi_2)\sin\omega_o t\ + \\ & \frac{1}{2}\,[\,\phi_3^{\,2} - \phi_1^{\,2} + 2\,\theta'(\phi_3 - \phi_1)\,]\sin\omega_o t\ + \\ & \frac{1}{2}\,[\,\phi_4^{\,2} - \phi_2^{\,2} + 2\,\theta'(\phi_4 - \phi_2)\,]\cos\omega_o t + O(\phi_i^{\,3})\end{aligned}$$

Appendix F — To find the inverse matrix $[B]^{-1}$ for inplane orthogonal transformation,

$$
[B] = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 & -1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & -\frac{4b}{c} \\ 0 & 1 & 0 & 0 & F & -G & 0 \\ 0 & 0 & 1 & 0 & -G & -F & 0 \end{bmatrix}
\xrightarrow[\substack{C_5 - FC_2 \\ C_6 + GC_2}]{C_7 + \frac{4b}{c} C_1}
\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 & -1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -G & -F & 0 \end{bmatrix}
$$

$$
\xrightarrow[C_6 + FC_3]{C_5 + GC_3}
\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 & -1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}
\xrightarrow[\frac{C_7 - C_5}{2}]{\frac{C_6 - C_4}{2}}
\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 & -1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
\xrightarrow[C_4 + C_6]{C_5 + C_7}
\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}
\xrightarrow[\frac{C_6 + C_7}{2}]{\frac{C_4 + C_5}{2}}
\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
\xrightarrow[\substack{C_5 - C_4 \\ C_7 - C_6}]{}
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}
\xrightarrow[\substack{C_1 \longleftrightarrow C_4 \\ C_2 \longleftrightarrow C_5 \\ C_3 \longleftrightarrow C_7 \\ C_4 \longleftrightarrow C_6 \\ C_5 \longleftrightarrow C_6}]{}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

$$
\xrightarrow[\substack{C_7 + \frac{4b}{c} C_1 \\ C_5 - FC_2 \\ C_6 + GC_2}]{}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \frac{4b}{c} \\
0 & 1 & 0 & 0 & -F & G & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\xrightarrow[\substack{C_5 + GC_3 \\ C_6 + FC_3}]{}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \frac{4b}{c} \\
0 & 1 & 0 & 0 & -F & G & 0 \\
0 & 0 & 1 & 0 & G & F & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
\xrightarrow[\substack{\frac{C_6 - C_4}{2} \\ \frac{C_7 - C_5}{2}}]{}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \frac{2b}{c} \\
0 & 1 & 0 & 0 & -F & \frac{G}{2} & \frac{F}{2} \\
0 & 0 & 1 & 0 & G & \frac{F}{2} & -\frac{G}{2} \\
0 & 0 & 0 & 1 & 0 & -\frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}
\end{pmatrix}
\xrightarrow[\substack{C_5 + C_7 \\ C_4 + C_6}]{}
\begin{pmatrix}
1 & 0 & 0 & 0 & \frac{2b}{c} & 0 & \frac{2b}{c} \\
0 & 1 & 0 & \frac{G}{2} & -\frac{F}{2} & \frac{G}{2} & \frac{F}{2} \\
0 & 0 & 1 & \frac{F}{2} & \frac{G}{2} & \frac{F}{2} & -\frac{G}{2} \\
0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\
0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2}
\end{pmatrix}
$$

$$
\xrightarrow[\substack{\frac{C_4 + C_5}{2} \\ \frac{C_6 + C_7}{2}}]{}
\begin{bmatrix}
1 & 0 & 0 & b/c & 2b/c & b/c & 2b/c \\
0 & 1 & 0 & (G-F)/4 & -F/2 & (G+F)/4 & F/2 \\
0 & 0 & 1 & (G-F)/4 & G/2 & (F-G)/4 & -G/2 \\
0 & 0 & 0 & 1/4 & 0 & -1/4 & 0 \\
0, & 0 & 0 & 1/4 & 1/2 & -1/4 & -1/2 \\
0 & 0 & 0 & 1/4 & 0 & 1/4 & 0 \\
0 & 0 & 0 & 1/4 & 1/2 & 1/4 & 1/2
\end{bmatrix}
$$

$$
\xrightarrow[\substack{C_5 - C_4 \\ C_7 - C_6}]{}
\begin{bmatrix}
1 & 0 & 0 & b/c & b/c & b/c & b/c \\
0 & 1 & 0 & (G-F)/4 & -(G+F)/4 & (G+F)/4 & (F-G)/4 \\
0 & 0 & 1 & (G+F)/4 & (G-F)/4 & (F-G)/4 & -(F+G)/4 \\
0 & 0 & 0 & 1/4 & -1/4 & -1/4 & 1/4 \\
0 & 0 & 0 & 1/4 & 1/4 & -1/4 & -1/4 \\
0 & 0 & 0 & 1/4 & -1/4 & 1/4 & -1/4 \\
0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4
\end{bmatrix}
$$

$$
\xrightarrow[\substack{C_1 \longleftrightarrow C_4 \\ C_2 \longleftrightarrow C_5 \\ C_3 \longleftrightarrow C_7 \\ C_4 \longleftrightarrow C_6 \\ C_5 \longleftrightarrow C_6}]{}
\begin{bmatrix}
b/c & b/c & b/c & b/c & 1 & 0 & 0 \\
(G-F)/4 & -(G+F)/4 & (F-G)/4 & (F+G)/4 & 0 & 1 & 0 \\
(G+F)/4 & (G-F)/4 & -(F+G)/4 & (F-G)/4 & 0 & 0 & 1 \\
1/4 & -1/4 & 1/4 & -1/4 & 0 & 0 & 0 \\
1/4 & 1/4 & -1/4 & -1/4 & 0 & 0 & 0 \\
1/4 & -1/4 & -1/4 & 1/4 & 0 & 0 & 0 \\
1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\begin{array}{c}
R_1/\alpha \\ R_2/\mu \\ R_3/\mu \\ R_4/\beta \\ R_5/\nu \\ R_6/\nu \\ R_7/\gamma
\end{array}
\begin{bmatrix}
b/c\alpha & b/c\alpha & b/c\alpha & b/c\alpha & 1/\alpha & 0 & 0 \\
(G-F)/4\mu & -(G+F)/4\mu & (F-G)/4\mu & (F+G)/4 & 0 & 1/\mu & 0 \\
(G+F)/4\mu & (G-F)/4\mu & -(F+G)/4\mu & (F-G)/4\mu & 0 & 0 & 1/\mu \\
1/4\beta & -1/4\beta & 1/4\beta & 1/4\beta & 0 & 0 & 0 \\
1/4\nu & 1/4\nu & -1/4\nu & -1/4\nu & 0 & 0 & 0 \\
1/4\nu & -1/4\nu & -1/4\nu & -1/4\nu & 0 & 0 & 0 \\
1/4\gamma & 1/4\gamma & 1/4\gamma & 1/4\gamma & 0 & 0 & 0
\end{bmatrix}
$$

$$[\,B\,] \xrightarrow[\substack{C_1/n_1 \\ C_{2,3}/n_2 \\ C_{4,5}/n_3 \\ C_6/n_4 \\ C_7/n_5}]{} \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -\frac{2b}{c} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{2b}{c} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 4d/\mathfrak{m} \end{bmatrix}$$

$$\xrightarrow[\substack{C_4+\frac{2b}{c}C_3 \\ C_5+\frac{2b}{c}C_2 \\ C_7-\frac{4d}{\mathfrak{m}}C_1}]{} \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\substack{(C_7-C_6)/2 \\ C_6+C_7}]{} \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[\substack{(C_6+C_4)/2 \\ (C_7+C_5)/2}]{} \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\substack{C_4-C_6 \\ C_5-C_7}]{} \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$
\xrightarrow[\substack{C_1 \longleftrightarrow C_4 \\ C_2 \longleftrightarrow C_5 \\ C_3 \longleftrightarrow C_6 \\ C_4 \longleftrightarrow C_7 \\ C_5 \longleftrightarrow C_6}]{}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\xrightarrow[\substack{C_1/n_1 \\ C_{2,3}/n_2 \\ C_{4,5}/n_3 \\ C_6/n_4 \\ C_7/n_5}]{}
\begin{pmatrix}
1/n_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/n_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/n_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/n_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/n_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1/n_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/n_5
\end{pmatrix}
$$

$$
\xrightarrow[\substack{C_4 + \frac{2b}{c} C_3 \\ C_5 + \frac{2b}{c} C_2 \\ C_7 - \frac{4d}{m} C_1}]{}
\begin{pmatrix}
1/n_1 & 0 & 0 & 0 & 0 & 0 & -4d/m \\
0 & 1/n_2 & 0 & 0 & 2b/cn_2 & 0 & 0 \\
0 & 0 & 1/n_2 & 2b/cn_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/n_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/n_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1/n_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/n_5
\end{pmatrix}
$$

$$
\xrightarrow[\substack{(C_7 - C_6)/2 \\ C_6 + C_7}]{}
\begin{pmatrix}
1/n_1 & 0 & 0 & 0 & 0 & -2d/m n_1 & -2d/m n_1 \\
0 & 1/n_2 & 0 & 0 & 2b/cn_2 & 0 & 0 \\
0 & 0 & 1/n_3 & 2b/cn_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/n_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/n_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1/2n_4 & -1/2n_4 \\
0 & 0 & 0 & 0 & 0 & 1/2n_5 & 1/2n_5
\end{pmatrix}
$$

$$
\xrightarrow[\substack{(C_6 + C_4)/2 \\ (C_7 + C_5)/2}]{}
\begin{pmatrix}
1/n_1 & 0 & 0 & 0 & 0 & -d/\eta n_1 & -d/\eta n_1 \\
0 & 1/n_2 & 0 & 0 & 2b/cn_2 & 0 & b/cn_2 \\
0 & 0 & 1/n_3 & 2b/cn_2 & 0 & b/cn_2 & 0 \\
0 & 0 & 0 & 1/n_3 & 0 & 1/2n_3 & 0 \\
0 & 0 & 0 & 0 & 1/n_3 & 0 & 1/2n_3 \\
0 & 0 & 0 & 0 & 0 & 1/4n_4 & -1/4n_4 \\
0 & 0 & 0 & 0 & 0 & 1/4n_5 & 1/4n_5
\end{pmatrix}
$$

$$
\xrightarrow[\substack{C_4 - C_6 \\ C_5 - C_7}]{}
\begin{pmatrix}
1/n_1 & 0 & 0 & d/\eta n_1 & d/\eta n_1 & -d/\eta n_1 & -d/\eta n_1 \\
0 & 1/n_2 & 0 & 0 & b/cn_2 & 0 & b/cn_2 \\
0 & 0 & 1/n_2 & b/cn_2 & 0 & b/cn_2 & 0 \\
0 & 0 & 0 & 1/2n_3 & 0 & 1/2n_3 & 0 \\
0 & 0 & 0 & 0 & 1/2n_3 & 0 & 1/2n_3 \\
0 & 0 & 0 & -1/4n_4 & 1/4n_4 & 1/4n_4 & -1/4n_4 \\
0 & 0 & 0 & -1/4n_5 & -1/4n_5 & 1/4n_5 & 1/4n_5
\end{pmatrix}
$$

$$
\xrightarrow[\substack{C_1 \longleftrightarrow C_4 \\ C_2 \longleftrightarrow C_5 \\ C_3 \longleftrightarrow C_6 \\ C_4 \longleftrightarrow C_7 \\ C_5 \longleftrightarrow C_6}]{}
\begin{pmatrix}
d/\eta n_1 & d/\eta n_1 & -d/\eta n_1 & -d/\eta n_1 & 0 & 0 & 1/n_1 \\
0 & b/cn_2 & 0 & b/cn_2 & 0 & 1/n_2 & 0 \\
b/cn_2 & 0 & b/cn_2 & 0 & 1/n_2 & 0 & 0 \\
1/2n_3 & 0 & 1/2n_3 & 0 & 0 & 0 & 0 \\
0 & 1/2n_3 & 0 & 1/2n_3 & 0 & 0 & 0 \\
-1/4n_4 & 1/4n_4 & 1/4n_4 & -1/4n_4 & 0 & 0 & 0 \\
-1/4n_5 & -1/4n_5 & -1/4n_5 & 1/4n_5 & 0 & 0 & 0
\end{pmatrix}
$$

Appendix H Inverse Orthonormal Matrix $[\,B\,]^{-1}$ for the Case of Unequal Length Boom Pairs with Translation

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & -1 & \mu_- & \mu_+ \\
0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & \mu_- & \mu_+ \\
0 & 0 & 0 & -1 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & \nu_+ & \nu_- \\
0 & 1 & 0 & f_1 & -g & 0 & 0 \\
0 & 0 & 1 & g_1 & f & 0 & 0
\end{pmatrix}
\xrightarrow[\substack{C_7 - \nu_- C_1 \\ C_6 - \nu_+ C_1 \\ C_5 + g\, C_2 \\ C_4 - f_1 C_2}]{}
\begin{pmatrix}
0 & 0 & 0 & 0 & -1 & \mu_- & \mu_+ \\
0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & \mu_- & \mu_+ \\
0 & 0 & 0 & -1 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & g_1 & f & 0 & 0
\end{pmatrix}
$$

$$
\xrightarrow[\substack{C_4 - g_1 C_3 \\ C_5 - f\, C_3 \\ C_6 + C_4 \\ C_7 + C_4}]{}
\begin{pmatrix}
0 & 0 & 0 & 0 & -1 & \mu_- & \mu_+ \\
0 & 0 & 0 & 1 & 0 & 2 & 2 \\
0 & 0 & 0 & 0 & 1 & \mu_- & \mu_+ \\
0 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}
\xrightarrow[\substack{C_6 - \mu_- C_5 \\ C_7 - \mu_+ C_5}]{}
\begin{pmatrix}
0 & 0 & 0 & 0 & -1 & 2\mu_- & 2\mu_+ \\
0 & 0 & 0 & 1 & 0 & 2 & 2 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
\xrightarrow[\substack{C_6/2 \\ C_7/2 \\ C_7 - C_6}]{}
\begin{pmatrix}
0 & 0 & 0 & 0 & -1 & \mu_- & \mu_+ - \mu_- \\
0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}
\xrightarrow[\substack{C_7/(\mu_+ - \mu_-) \\ C_6 - \mu_- C_7 \\ C_5 + C_7 \\ C_4 - C_6 \\ -C_4}]{}
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$\xrightarrow[\substack{C_1 \leftrightarrow C_7 \\ C_2 \leftrightarrow C_6 \\ C_3 \leftrightarrow C_5 \\ C_5 \leftrightarrow C_7}]{} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[\substack{C_7 - \nu_- C_1 \\ C_6 - \nu_- C_1 \\ C_5 + g\, C_2 \\ C_4 - f_1 C_2}]{} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -\nu_+ & -\nu_- \\ 0 & 1 & 0 & -f_1 & g & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[\substack{C_4 - g_1 C_3 \\ C_5 - f\, C_3 \\ C_6 + C_4 \\ C_7 + C_4}]{} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -\nu_+ & -\nu_- \\ 0 & 1 & 0 & -f_1 & g & -f_1 & -f_1 \\ 0 & 0 & 1 & -g_1 & -f & -g_1 & -g_1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[\substack{C_6 - \mu_- C_5 \\ C_7 - \mu_+ C_5}]{} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -\nu_+ & -\nu_- \\ 0 & 1 & 0 & -f_1 & g & -\mu_- g - f_1 & -\mu_+ g - f_1 \\ 0 & 0 & 1 & -g_1 & -f & \mu_- f - g_1 & \mu_+ f - g_1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -\mu_- & -\mu_+ \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[\substack{C_6/2\\ C_7/2\\ C_7-C_6}]{} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -\nu_+/2 & (\nu_+-\nu_-)/2 \\ 0 & 1 & 0 & -f_1 & g & -(\mu_-g+f_1)/2 & (\mu_- -\mu_+)g/2 \\ 0 & 0 & 1 & -g_1 & -f & (\mu_-f-g_1)/2 & (\mu_+-\mu_-)f/2 \\ 0 & 0 & 0 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\mu_-/2 & (\mu_- -\mu_+)/2 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \end{pmatrix}$$

$$\xrightarrow[\substack{C_7/(\mu_+-\mu_-)\\ C_6-\mu_-C_7\\ C_5+C_7}]{} \begin{pmatrix} 1 & 0 & 0 & 0 & (\nu_+-\nu_-)/\zeta & (\mu_-\nu_- -\mu_+\nu_+)/\zeta & (\nu_+-\nu_-)/\zeta \\ 0 & 1 & 0 & -f_1 & g/2 & -f_1/2 & -g/2 \\ 0 & 0 & 1 & -g_1 & -f/2 & -g_1/2 & f/2 \\ 0 & 0 & 0 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 0 & 0 & -1/\zeta & \mu_+/\zeta & -1/\zeta \\ 0 & 0 & 0 & 0 & 1/\zeta & -\mu_-/\zeta & 1/\zeta \end{pmatrix}$$

$$\xrightarrow[\substack{C_4-C_6\\ (-)\,C_4}]{} \begin{pmatrix} 1 & 0 & 0 & (\mu_-\nu_- -\mu_+\nu_+)/\zeta & (\nu_+-\nu_-)/\zeta & (\mu_-\nu_- -\mu_+\nu_+)/\zeta & (\nu_+-\nu_-)/\zeta \\ 0 & 1 & 0 & f_1/2 & g/2 & -f_1/2 & -g/2 \\ 0 & 0 & 1 & g_1/2 & -f/2 & -g_1/2 & f/2 \\ 0 & 0 & 0 & -1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 0 & \mu_+/\zeta & -1/\zeta & \mu_+/\zeta & -1/\zeta \\ 0 & 0 & 0 & -\mu_-/\zeta & 1/\zeta & -\mu_-/\zeta & 1/\zeta \end{pmatrix}$$

$$\xrightarrow[\substack{C_1\leftrightarrow C_7\\ C_2\leftrightarrow C_6\\ C_3\leftrightarrow C_5\\ C_5\leftrightarrow C_7}]{} \begin{bmatrix} (\nu_+-\nu_-)/\zeta & (\mu_-\nu_- - \mu_+\nu_+)/\zeta & (\nu_+-\nu_-)/\zeta & (\mu_-\nu_- - \mu_+\nu_+)/\zeta & 1 & 0 & 0 \\ -g/2 & -f_1/2 & g/2 & f_1/2 & 0 & 1 & 0 \\ f/2 & -g_1/2 & -f/2 & g_1/2 & 0 & 0 & 1 \\ 0 & 1/2 & 0 & -1/2 & 0 & 0 & 0 \\ -1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ -1/\zeta & \mu_+/\zeta & -1/\zeta & \mu_+/\zeta & 0 & 0 & 0 \\ 1/\zeta & -\mu_-/\zeta & 1/\zeta & -\mu_-/\zeta & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[\substack{R_i\,/\,n_i\\ (i=1,\ldots 7)}]{} \begin{bmatrix} (\nu_+-\nu_-)/\zeta n_1 & (\mu_-\nu_- - \nu_+\mu_+)/\zeta n_1 & (\nu_+-\nu_-)/\zeta n_1 & (\mu_-\nu_- - \mu_+\nu_+)/\zeta n_1 & n_1^{-1} & 0 & 0 \\ -g/2n_2 & -f_1/2n_2 & g/2n_2 & f_1/2n_2 & 0 & n_2^{-1} & 0 \\ f/2n_3 & -g_1/2n_3 & -f/2n_3 & g_1/2n_3 & 0 & 0 & n_3^{-1} \\ 0 & 1/2n_4 & 0 & -1/2n_4 & 0 & 0 & 0 \\ -1/2n_5 & 0 & 1/2n_5 & 0 & 0 & 0 & 0 \\ -1/\zeta n_6 & \mu_+/\zeta n_6 & -1/\zeta n_6 & \mu_+/\zeta n_6 & 0 & 0 & 0 \\ 1/\zeta n_7 & -\mu_-/\zeta n_7 & 1/\zeta n_7 & -\mu_-/\zeta n_7 & 0 & 0 & 0 \end{bmatrix}$$

Where $\zeta = 2(\mu_+ - \mu_-)$

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